A new regularization of loop integral
A new look on the hirerarchy problem

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OUTLINE:

- Motivation
  - Hierarchy problem in loop integral
  - Bose-Einstein condensation
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- Discrete regularization of loop function
  - Regularization and comparison with Dimensional regularization
  - Two level of understanding of the new regularization
- Implication of new regularization
  - Predications in the QED
  - Hierarchy problem
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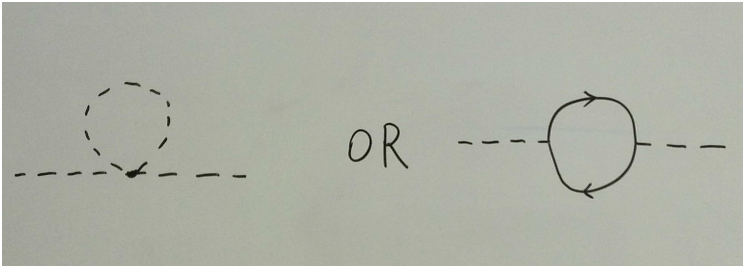
Motivation

Hierarchy problem in loop integral:

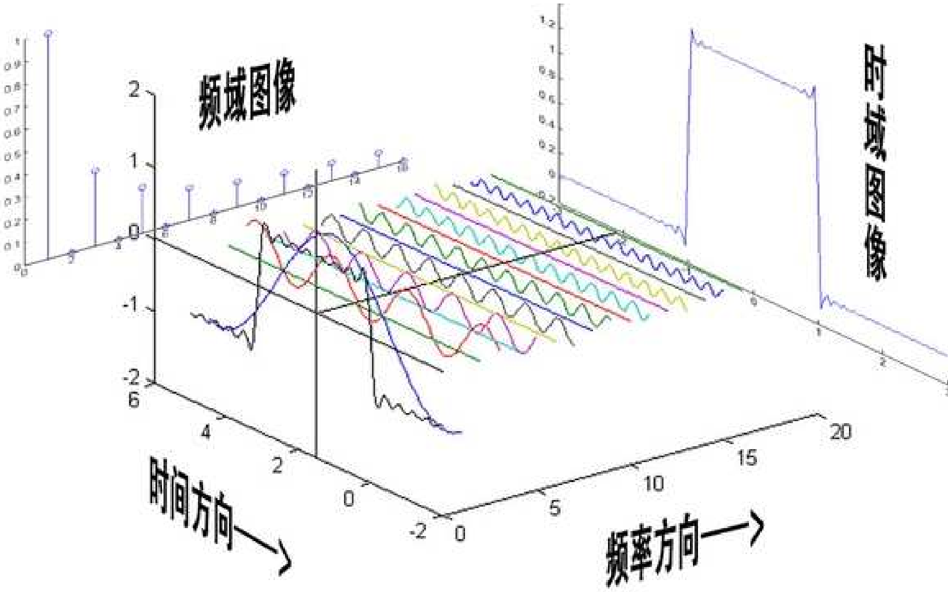


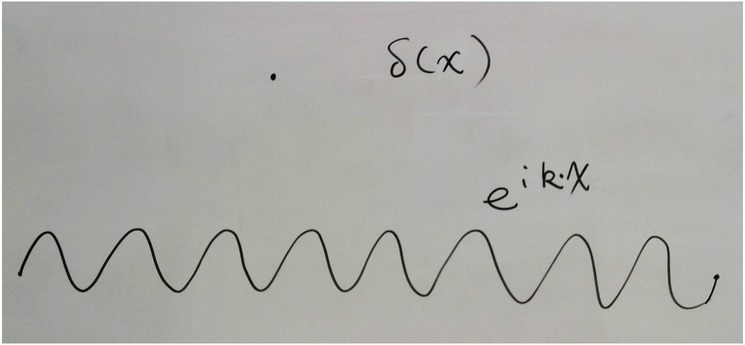
$$\int \frac{d^d l}{(2\pi)^d} \frac{i}{l^2 - m^2 + i\epsilon} \sim \Lambda^2$$

Quadratically divergent !

It is quit obscure to do such an integral since in general the energy of a particle in the quantum world have no reason to be continuous.

Fourian transformation between time and frequency



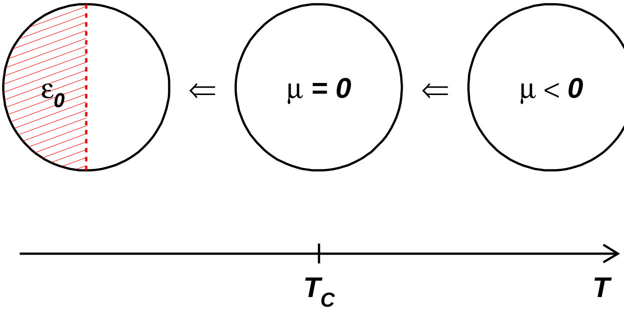


- A "point" or a "plane wave" is only concept in mathematical QFT.
- The divergence is kind of "phase transition"?
- Can the divergence be removed by Riemann ζ function?

We should be very careful about the operation from a summing to a integral

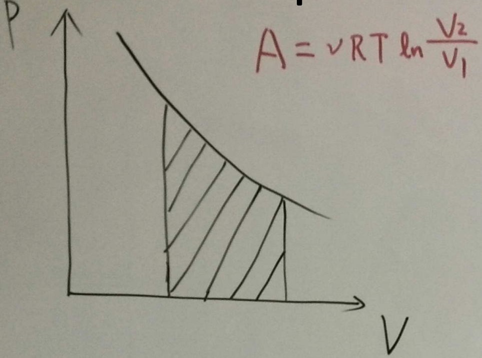
Bose Einstein Condensation:

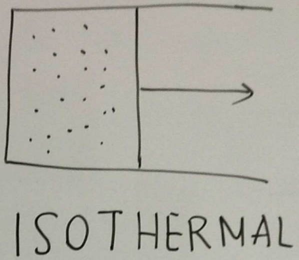
$$a_l = \frac{\omega_l}{e^{\frac{\epsilon_l - \mu}{kT}} - 1}$$

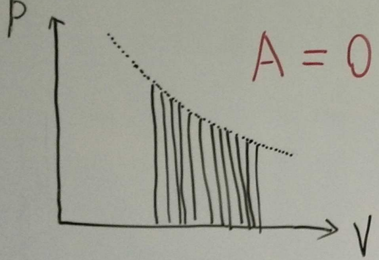


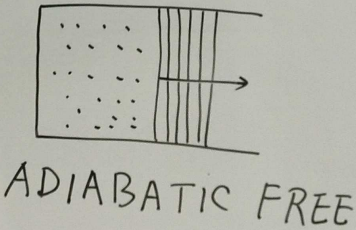
The way from discrete summing to integral may go wrong in some physical systems.

work of thermal process:









Riemann ζ function:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int \mathrm{d}t \frac{t^{s-1}}{e^t - 1} = \sum_{n=1}^{\infty} n^{-s}$$

.

$$\zeta(-1) = 1 + 2 + 3 + 4 + ... = -\frac{1}{12}$$

A brief Proof

$$\begin{aligned} \zeta(-1) &= \frac{\zeta(-1) - (2\zeta(-1) - \sum_{n=1}^{\infty} 1)}{2} \\ \rightarrow \zeta(-1) &= \frac{\sum_{n=1}^{\infty} 1}{3} \end{aligned}$$

Taylor expansion of $\frac{1}{(1+x)^2}$ at $x = 0$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + ...$$

$x = 1 - \epsilon$ then $\sum_{n=1}^{\infty} 1 = -\frac{1}{4}$ then $\zeta(-1) = -\frac{1}{12}$

Discrete regularization

Procedure of Dimensional Regularization

$$\frac{i}{16\pi^2}B_{\mu\nu}(p,m_1^2,m_2^2) \;=\; \int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{k_\mu k_\nu}{[k^2-m_1^2+i\epsilon][(k-p)^2-m_2^2+i\epsilon]}\,,$$

- Feynman parameterization

use

$$\frac{1}{AB}=\int_0^1 \mathrm{d}x\mathrm{d}y\delta(x+y-1)\frac{1}{[xA+yB]^2}$$

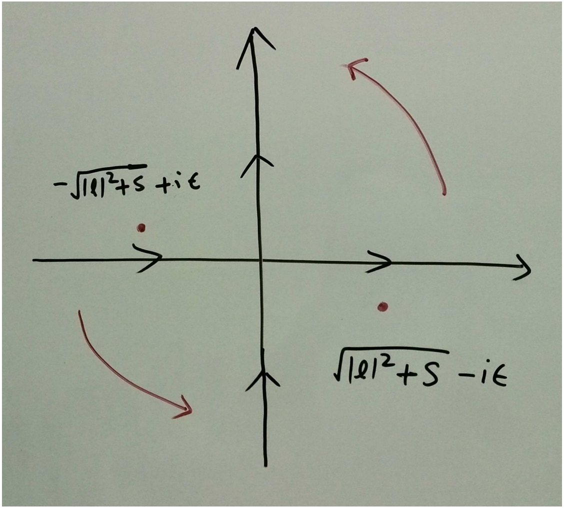
and let $l=k-xp$

$$B \;=\; \int\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}x\frac{l_\mu l_\nu+x^2p_\mu p_\nu}{[l^2-S(x)]^2}$$

in which

$$S(x)\equiv p^2x^2-(p^2+m_1^2-m_2^2)x+m_1^2$$

- Wick Rotation



Rotate the integral from Minkovski space to Euclidean space:

$$I(d,n,S) = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2+S)^n}$$

- Dimensional Regularization

$$
\begin{aligned}
I_D(d,n,S,\mu) &= \mu^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2+S)^n} \\
&= \mu^\epsilon \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{S}\right)^{n-\frac{d}{2}}
\end{aligned}
$$

1. $d = 4-\epsilon$, $n = 1, 2$

$$
\Gamma(-1+\frac{\epsilon}{2}), \quad \Gamma(\frac{\epsilon}{2})
$$

   divergent

2. $d = 3-\epsilon$, $n = 1, 2$ the integral is finite. interestingly

$$
I_D(d,n,S,\mu) = \mu^\epsilon \int \frac{d^{3-\epsilon} l}{(2\pi)^{3-\epsilon}} \frac{1}{l^2+S}
$$

   superficially divergent, but finite in complex integral.

Discrete regularization

$$I(d,n,S) = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2+S)^n}$$

we consider the virtual particle like an oscillator, which energy gap is denoted as l_0, the energy level is jl_0.

$$\begin{aligned} I_W(d,n,S,l_0) &= \frac{l_0}{2\pi} \int \frac{d^{d-1} l}{(2\pi)^{d-1}} \frac{1}{(l^2+S)^n} \\ &+ \frac{l_0}{\pi} \sum_{j=1}^{\infty} \int \frac{d^{d-1} l}{(2\pi)^{d-1}} \frac{1}{(l^2+j^2 l_0^2+S)^n} \end{aligned}$$

Then

$$\begin{aligned} I_W &= \frac{l_0}{2\pi} \frac{1}{(4\pi)^{d/2-1/2}} \frac{\Gamma(n-\frac{d}{2}+\frac{1}{2})}{\Gamma(n)} \left(\frac{1}{S}\right)^{n-\frac{d}{2}+\frac{1}{2}} \\ &+ \frac{l_0}{\pi} \sum_{j=1}^{\infty} \frac{1}{(4\pi)^{d/2-1/2}} \frac{\Gamma(n-\frac{d}{2}+\frac{1}{2})}{\Gamma(n)} \left(\frac{1}{j^2 l_0^2+S}\right)^{n-\frac{d}{2}+\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
I_W &= \frac{l_0}{\pi}\frac{1}{(4\pi)^{d/2-1/2}}\frac{\Gamma(n-\frac{d}{2}+\frac{1}{2})}{\Gamma(n)}\left[\frac{1}{2}\left(\frac{1}{S}\right)^{n-\frac{d}{2}+\frac{1}{2}}+\sum_{j=1}^{\infty}\left(j^2 l_0^2+S\right)^{-(n-\frac{d}{2}+\frac{1}{2})}\right], \\
&= \frac{l_0^{-2n+d}}{\pi}\frac{1}{(4\pi)^{d/2-1/2}}\frac{\Gamma(n-\frac{d}{2}+\frac{1}{2})}{\Gamma(n)}\left[\frac{1}{2}(S/l_0^2)^{-(n-\frac{d}{2}+\frac{1}{2})}+\sum_{j=1}^{\infty}\left(j^2+\frac{S}{l_0^2}\right)^{-(n-\frac{d}{2}+\frac{1}{2})}\right], \\
&= \frac{4l_0^{-2n+d}}{(4\pi)^{d/2+1/2}}\frac{\Gamma(n-\frac{d}{2}+\frac{1}{2})}{\Gamma(n)}\left[E_1^{S/l_0^2}(n-\frac{d}{2}+\frac{1}{2};\ 1)+\frac{1}{2}(S/l_0^2)^{-(n-\frac{d}{2}+\frac{1}{2})}\right]
\end{aligned}
$$

All the divergence from the Γ function now vanish in case of even number dimension. The divergences are absorbed by the Epstein-Hurwitz function :

$$
E_1^{c^2}(s;\ 1) \equiv \sum_{j=1}^{\infty}(j^2+c^2)^{-s}
$$

where $c^2 = S/l_0^2$ and $s = n - d/2 + 1/2$.

Epstein-Hurwitz function can be regulated by Riemann ζ function in case of $c^2 \leq 1$, the results depend on the parameter s, which is:

1. in case of $\frac{1}{2} - s \in N$:

$$E_1^{c^2}(s;1) = -\frac{(-1)^{-(s-1/2)}\pi^{1/2}}{2\Gamma(s)\Gamma(\frac{3}{2}-s)}c^{1-2s}\left[\psi(\frac{1}{2}) - \psi(\frac{3}{2}-s) + \ln c^2 + 2\gamma\right] - \frac{1}{2}c^{-2s},$$
$$- \sum_{k=0, k\neq \frac{1}{2}-s}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k!\Gamma(s)}\zeta(2k+2s)c^{2k}.$$

2. in case of $\frac{1}{2} - s \notin N$ and $-s \notin N$:

$$E_1^{c^2}(s;1) = \frac{\pi^{1/2}}{2\Gamma(s)}\Gamma(s-\frac{1}{2})c^{1-2s} - \frac{1}{2}c^{-2s},$$
$$-\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(k+s)}{k!\Gamma(s)}\zeta(2k+2s)c^{2k}.$$

3. in case of $-s \in N$:

$$E_1^{c^2}(s;1) = -\sum_{k=0}^{-s}(-1)^k\frac{\Gamma(k+s)}{k!\Gamma(s)}\zeta(2k+2s)c^{2k}.$$

where N is the natural number $N = 0, 1, 2, 3, ...$

$d = 4$

1. $n = 1$

$$s = n - \frac{d}{2} + \frac{1}{2} = -\frac{1}{2}, \quad \frac{1}{2} - s = 1$$

2. $n = 2$

$$s = n - \frac{d}{2} + \frac{1}{2} = \frac{1}{2}, \quad \frac{1}{2} - s = 0$$

3. $n \geq 3$

$$\frac{1}{2} - s \notin N, \quad - s \notin N$$

$E_1^{c^2}(s;\ 1)$ is a continuous function in the complex plane.

$$\lim_{s \to -\frac{1}{2}} E_1^{c^2}(s; 1) = E_1^{c^2}(-\frac{1}{2}; 1)$$

All the divergences vanish in our new method of regularization. left only with two kinds of terms:

1. finite term composed by the product of Γ functions
2. a summation of a power series of S/l_0^2.

Comparision with of DR

1. DR

$$B_0^D = \triangle + \ln\frac{\mu^2}{m_1^2} - \int_0^1 \mathrm{d}x \ln S(x)\,,$$

in which $\triangle = \frac{2}{\epsilon} - \gamma + \ln 4\pi$

2. WWZ

$$B_0^W = 2\ln 2 - 2\gamma + \ln\frac{l_0^2}{m_1^2} - \int_0^1 \mathrm{d}x \ln S(x)\,,$$
$$-2\sum_{k=1}^{\infty}(-1)^k \frac{\Gamma(k+1/2)}{k!\pi^{1/2}}\zeta(2k+1)\int_0^1 \mathrm{d}x \left(\frac{m_1^2}{l_0^2}S(x)\right)^k .$$

All the other functions are similar !

$$\frac{i}{16\pi^2} B_{\mu\nu}(p, m_1^2, m_2^2) = \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{[l^2 - m_1^2 + i\epsilon][(l-p)^2 - m_2^2 + i\epsilon]} \,,$$

$$\begin{aligned} B^D_{\mu\nu} \;=\; & \frac{1}{3}\Big\{ p_\mu p_\nu \Big[\Delta + \ln\frac{\mu^2}{m_1^2} - 3\int_0^1 \mathrm{d}x x^2 \ln S(x) \Big] \\ & + \frac{g_{\mu\nu}}{d} \Big[(3m_1^2 + 3m_2^2 - p^2)(\Delta + \ln\frac{\mu^2}{m_1^2} + 1) \\ & - \frac{3m_1^2 + 3m_2^2 - p^2}{2} - 6m_1^2 \int_0^1 \mathrm{d}x S(x) \ln S(x) \Big] \Big\} \,, \end{aligned}$$

$$
\begin{aligned}
B^{W}_{\mu\nu} \;=\; & \frac{1}{3}\Bigg\{p_\mu p_\nu\left[2\ln 2-2\gamma+\ln\frac{l_0^2}{m_1^2}-3\int_0^1 \mathrm{d}x x^2\ln S(x)\right]\\
& +\frac{g_{\mu\nu}}{d}\Bigg[(3m_1^2+3m_2^2-p^2)(2\ln 2-2\gamma+\ln\frac{l_0^2}{m_1^2}+1)\\
& -\frac{3m_1^2+3m_2^2-p^2}{2}-6m_1^2\int_0^1 \mathrm{d}x S(x)\ln S(x)\Bigg]\\
& -6p_\mu p_\nu\Bigg[\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k+1/2)}{k!\pi^{1/2}}\zeta(2k+1)\\
& \times\int_0^1 \mathrm{d}x x^2\left(\frac{m_1^2}{l_0^2}S(x)\right)^k\Bigg]-6\frac{g_{\mu\nu}}{d}\Bigg[-l_0^2\sum_{k=0,k\neq 1}^{\infty}(-1)^k\\
& \times\frac{\Gamma(k-1/2)}{k!\pi^{1/2}}\zeta(2k-1)\int_0^1 \mathrm{d}x\left(\frac{m_1^2}{l_0^2}S(x)\right)^k+\sum_{k=1}^{\infty}(-1)^k\\
& \times\frac{\Gamma(k+1/2)}{k!\pi^{1/2}}\zeta(2k+1)\int_0^1 \mathrm{d}x\; m_1^2 S(x)\left(\frac{m_1^2}{l_0^2}S(x)\right)^k\Bigg]\Bigg\}\;.
\end{aligned}
$$

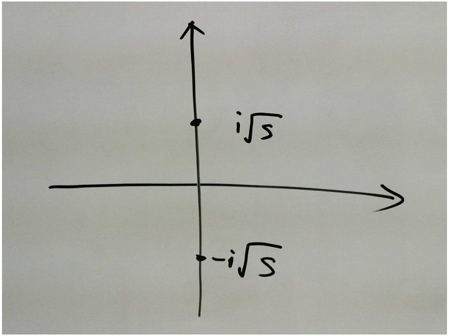
Two level of understanding our regularization:

1. Level I: **this method is a trick, by which we can get the almost the same results of dimensional regularization in case of $l_0^2 \gg S$.**

2. Level II: **What we are doing is an anti-BEC calculation, the divergences are in fact condensed in the vacuum. Then we should take a new look at the quantum field theory.**

Where does the divergence go ?

$$
\begin{aligned}
I_D(d,n,S,\mu) &= \mu^{\epsilon}\int \frac{d^{3-\epsilon}l}{(2\pi)^{3-\epsilon}}\frac{1}{l^2+S} \\
&= \mu^{\epsilon}\frac{1}{(4\pi)^{3/2}}\frac{\Gamma(-\frac{1-\epsilon}{2})}{\Gamma(1)}\left(\frac{1}{S}\right)^{-\frac{1-\epsilon}{2}} \to -\frac{\sqrt{S}}{4\pi}
\end{aligned}
$$



$$
I(d,n,S) = \int \frac{d^3l}{(2\pi)^3}\frac{1}{l^2+S} = \frac{1}{4\pi^2}2\pi i\frac{-S}{2i\sqrt{S}} = -\frac{\sqrt{S}}{4\pi}
$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int \mathrm{d}t \frac{t^{s-1}}{e^t - 1}$$

$$\zeta(-1) = \sum_{k=1}^{\infty} k = -\frac{1}{12}, \quad \zeta(0) = \sum_{k=1}^{\infty} 1 = -\frac{1}{2}$$

- $$\Gamma(-1 + \frac{\epsilon}{2}), \quad \Gamma(\frac{\epsilon}{2})$$

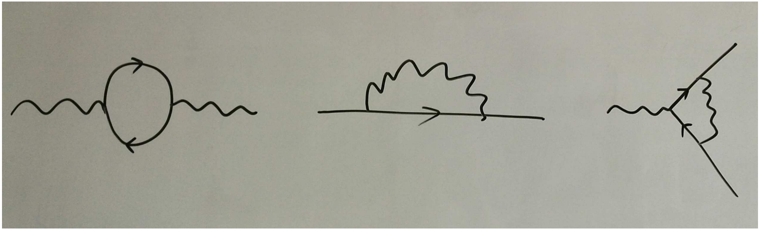
  are physics problems, divergence are removed by renormalization.

- $$\Gamma(-\frac{1}{2}), \quad \zeta(-1)$$

  are mathematics problems, divergence are regulated by mathematician.

Implications of the new regularization

Predications in the QED



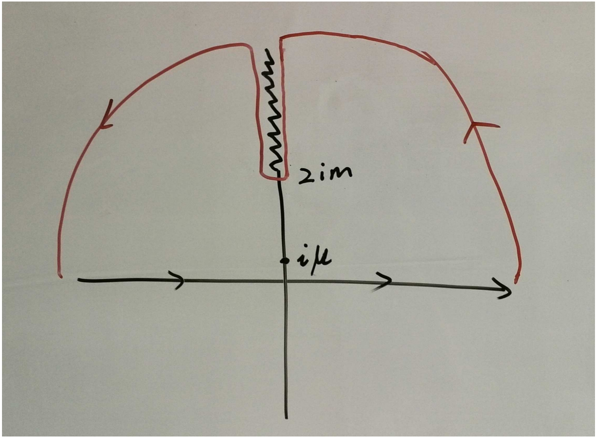
- *Electron magnetic movement a_e.*

$$\alpha_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi}\frac{\zeta(3)}{6}\frac{m_e^2}{l_0^2}\,.$$

- *Running of coupling strength $\alpha_{\text{eff}}(q^2)$.*

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln(\frac{-q^2}{A' m^2})}\,,$$

where $A' = \exp(\frac{5}{3} + \frac{\zeta(3)}{5})$.



- *Lamb shift.*
  Not changed by the new regularizations. The Uehling potential comes from the imaginary part of photon self energy $\hat{\Pi}_2(q^2)$ which not appear in the power series terms.

- *Gauge symmetry.*
  Ward identity requires:

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi(q^2)$$

Cut has additional term

$$e^2 \Lambda^2 g^{\mu\nu}$$

which violates the U(1) symmetry.
DR uses

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(2-\epsilon)\gamma^\nu$$

$$\ldots\ldots$$

protecting the symmetry.
The new regularization violates the gauge symmetry too. We can consider it as a auxilary method of DR, which means that we use DR to study the gauge symmetry and Lorentz symmetry. but use the WWZ to give the prediction of scalar function.

- *β function of the QED.*

  The energy scale l_0 is like the temperature of the vacuum, thus the β function of the coupling is kind of thermal capacitance of the a theory. Especially when the momentum approaches to the temperature then the β function will be exactly the capacitance: $(M^2 \to 1)$

$$\begin{aligned}\beta(\alpha) &= M\frac{\partial}{\partial M}(\text{counter terms}), \\ &= 2\frac{\partial}{\partial \ln M^2}(\text{counter terms}), \\ &\simeq 2\frac{\partial}{\partial M^2}(\text{counter terms}).\end{aligned}$$

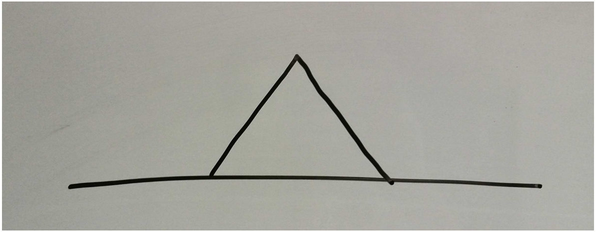
  β function of the QED is:

$$\beta(e) = \frac{\mathrm{e}^3}{12\pi^2} - \frac{1}{3}\zeta(3)\frac{\mathrm{e}^3}{16\pi^2}.$$

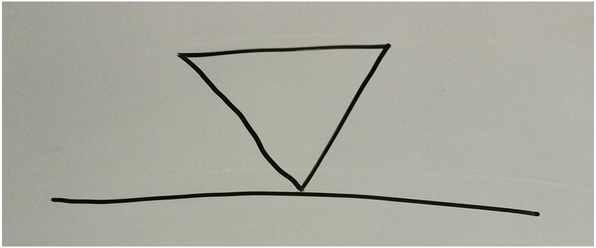
  The first term is the prediction of DR, the second term is the modification of the new regularization.

What is Hierarchy?

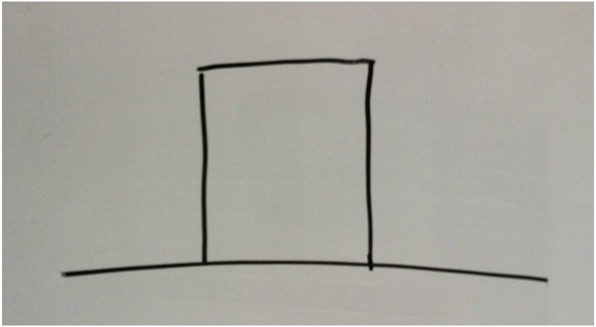
- Tuning with a symmetry and with a divergence



- Tuning without a symmetry but with a divergence



- Tuning without a symmetry and without divergence



Hierarchy problem of a scalar mass

1. *$\lambda\phi^4$ theory:* the leading term of mass counter term is

$$\delta_m = \frac{\lambda}{2}\frac{m^2}{16\pi^2}\left(2\ln 2 - 2\gamma + \ln\frac{l_0^2}{m^2} + 1 + \frac{l_0^2}{3m^2}\right) .$$

2. *Yukawa theory:* the leading term of mass counter term is

$$\begin{aligned}\delta_m \quad &= \quad -\frac{Y^2}{4\pi^2}\left[\frac{l_0^2}{3} + \int_0^1 dx(m_f^2 - x(1-x)m_s^2)\right.\\ &\left.\left(6\ln 2 - 6\gamma - 3\ln\frac{m_f^2 - x(1-x)m_s^2}{l_0^2} + 1\right)\right] + m_s^2\delta_Z ,\end{aligned}$$

$$\delta_Z = -\frac{3Y^2}{4\pi^2}\int_0^1 dx x(1-x)\left(2\ln 2 - 2\gamma - \frac{2}{3} - \ln\frac{m_f^2 - x(1-x)m_s^2}{l_0^2}\right) .$$

Points:

- All the physical variables are discrete. Continuous Lorentz symmetry is in fact conflict with Quantum Mechanics.
- "Point" QFT is only zero order approximation of real physics. Loop calculations must use discrete summation.
- A theory must be defined on an error scale $\Delta\mu$ not on an absolute scale μ. Integrated the heavy particles ($\Lambda \to \infty$) is inaccurate understanding of Quantum Mechanics.

Assumption:

1. l_0: energy gap, temperature of vacuum, or enery scale of a theory
2. jl_0: energy bound states, j is the quantum number,
3. $$\int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(l^2 + j^2 l_0^2 + S)^n}$$

   Distribution of jth bound states

We are tring to do a statistics of vacuum ?!

Conclusion

- **The divergence of a radiative correction is unphysical, emergence of divergence is because a wrong mathematical tools are used by physicists.**

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Thank you !