Commun. Theor. Phys. (Beijing, China) 41 (2004) pp. 335–338
© International Academic Publishers Vol. 41, No. 3, March 15, 2004

Solitary Wave and Wavelet*

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(Received June 6, 2003)

Abstract *In this paper, it is shown that the homoclinic orbits exist in iterated functional systems, so do the solitary wave structures. Moreover, Harr father wavelet, Mexican Cap wavelet, and other closed form wavelets have this solitary wave structure, too. So wavelet is a certain kind of solitary wave.*

PACS numbers: 02.90.+p
Key words: solitary waves, wavelet, homoclinic orbit, wavelet transform

1 Introduction

It is well known that wavelet transform[1−3] is a useful tool for multi-scale analysis. Because the mother wavelet is localized, so wavelet transform can identify the singularity of a piece of signal. Then there comes a question, i.e. whether a wavelet is a certain kind of solitary wave, since the solitary wave is also a kind of localized structure. To understand this question takes a lot of times. In the past, one took the opinion that the solitary wave could only exist in the nonlinear conservative systems, such as KdV equation, where the appearance of the solitary wave was a result of a balance between nonlinearity and dispersion. Later, the solitary wave is also found in the dissipative systems (see Ref. [4]), where its existence is the result of a balance between gain and loss of energy. So in order to form solitary wave in the dissipative systems, there must exist some regions where the energy is pumped inwards from an external source, as well as some regions where energy is dissipated outwards to the environment. Different from the solitary waves found in the nonlinear conservative or dissipative systems, the wavelets satisfy functional equation or ordinary differential equation (ODE) with variable coefficients, and they are all linear equations. In the following sections, we will show that the wavelet is also a certain kind of solitary wave.

2 Homoclinic Orbit in Iterated Functional System

Early in 1989, Hao pointed out[5] that the homoclinic orbit exists in iterated functional system. For example, the logistic map

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n) \quad n = 0, 1, 2, \ldots \tag{1}$$

as shown in Fig. 1. It is also called functional equation.[6,7]

Obviously, map (1) has two unstable fixed points $x_1^* = 0$ and $x_2^* = 3/4$. Starting from $x = 1/2$, twice iterations will arrive at the unstable fixed point $x_1^* = 0$, i.e., it leads to definite number sequence

$$\frac{1}{2}, 1, 0, 0, \ldots \tag{2}$$

In other words, the forward iteration falls into the unstable fixed point $x_1^* = 0$. The backward iterations, i.e. from x_{n+1} to x_n, return to the same point $x_1^* = 0$, too.

Therefore, the iterative sequence from 1/2 forms as follows:

$$\cdots, 0, 0, 1, \frac{1}{2}, 1, 0, 0, \cdots \tag{3}$$

This is a homoclinic orbit which approaches $x_1^* = 0$ in a limit with $n \to \pm\infty$. In Fig. 1, it is denoted by thick lines with arrows.

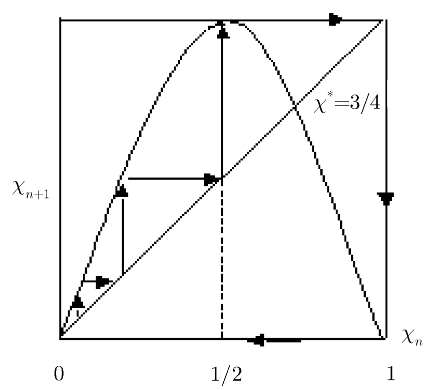

Fig. 1 The homoclinic orbit in map (1).

In the Refs. [8] ∼ [11], it was proposed that the solitary waves for partial differential equation (PDE) are corresponding to homoclinic orbits. Similarly, if the time variable n in (1) is replaced by wave variable $(j - n)$, where j denotes space localization, then the homoclinic orbit of map (1) will represent solitary wave.

3 Harr Wavelet in Wavelet Transform

The wavelet transform of a piece of signal $f(x)$ can be

*The project supported by National Natural Science Foundation of China under Grant Nos. 40045016 and No. 40175016
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written as[1−3,12]

$$T_g(a,t_0)=\frac{1}{a}\int_{-\infty}^{+\infty} f(x)g\Big(\frac{x-t_0}{a}\Big)\,\mathrm{d}t\,, \tag{4}$$

where $g((x-t_0)/a)$ is called mother wavelet. Because the argument of the variable $g((x-t_0)/a)$ is $(x-t_0)/a$, then $g((x-t_0)/a)$ can be written as a rightward travelling wave, $g(\xi)=g(x-ct)$, i.e., mother wavelet takes wave form.

For example, Harr wavelet can be written as

$$g(\xi)=\phi(2\xi)-\phi(2\xi-1)\,, \tag{5}$$

where $\phi(\xi)$ is called Harr father wavelet, or Harr scaling function. It satisfies scaling equation

$$\phi(\xi)=\phi(2\xi)+\phi(2\xi-1)\,. \tag{6}$$

Obviously, $\phi(\xi)=0$ is a fixed point of functional equation (6).

A solution to functional equation (6) is Harr father wavelet,

$$\phi(\xi)=\begin{cases}1, & 0\le\xi\le 1\,,\\ 0, & \text{otherwise}\,.\end{cases} \tag{7}$$

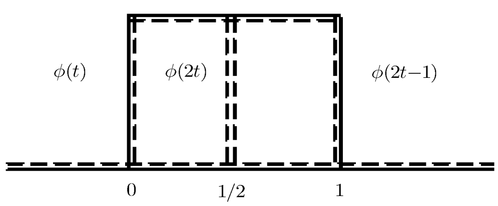

Fig. 2 Harr father wavelet, solitary wave.

Harr father wavelet is an isolated square wave, as shown in Fig. 2. From Fig. 2, we see that the scale of the first term $\phi(2\xi)$ on the right-hand side of Eq. (6) is the half as large as $\phi(\xi)$, and the second term $\phi(2\xi-1)$ is the result that $\phi(2\xi)$ shifts rightwards for half unit. From Fig. 2, we also see that $\phi(\xi)\to 0$, fixed point, when $\xi\to\pm\infty$.

The functional equations of the other wavelets, such as the scaling function equation for tent map and the quadratic Battle-lemarie scaling function, are

$$\phi(\xi)=\frac{1}{2}\phi(2\xi)+\phi(2\xi-1)+\frac{1}{2}\phi(2\xi-2)\,, \tag{8a}$$

$$\phi(\xi)=\frac{1}{4}\phi(2\xi)+\frac{3}{4}\phi(2\xi-1)+\frac{3}{4}\phi(2\xi-2)+\frac{1}{4}\phi(2\xi-3)\,. \tag{8b}$$

The solutions to Eq. (8) are all solitary waves with homoclinic point $\phi(\xi)=0$.

4 Mexican Cap Wavelet

Mexican Cap wavelet can be written as

$$g(\xi)=(1-\xi^2)\,\mathrm{e}^{-\xi^2/2}\,. \tag{9}$$

It is easily shown that $g(\xi)$ satisfies the following linear ODE with variable coefficients,[12]

$$g''(\xi)+\xi g'(\xi)+3g(\xi)=0\,, \tag{10}$$

where the super-primes of $g(\xi)$ denote derivative with respect to ξ.

Let

$$g(\xi)=\mathrm{e}^{-\xi^2/4}y(\xi)\,, \tag{11}$$

then equation (10) reduces to

$$y''(\xi)+\frac{1}{2}\Big(5-\frac{1}{2}\xi^2\Big)y(\xi)=0\,, \tag{12}$$

which is the well-known second kind of Weber equation.

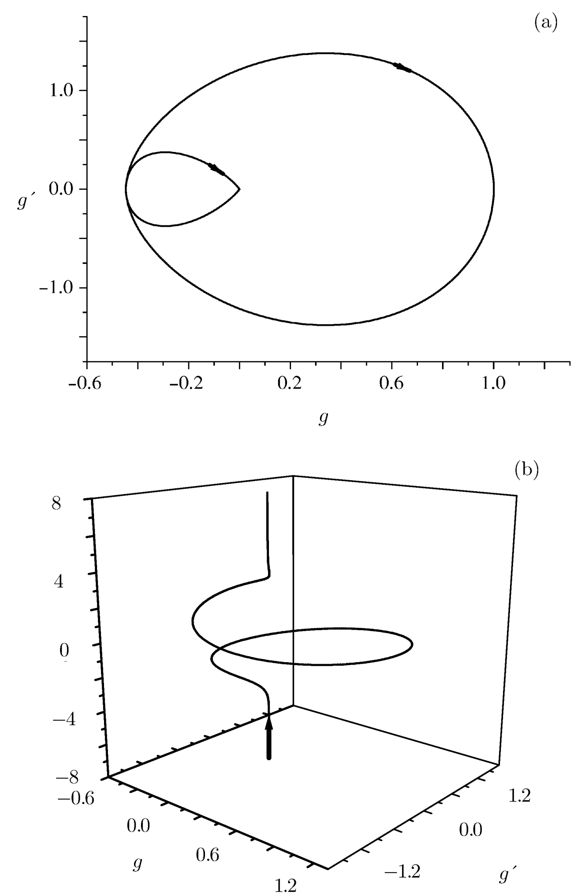

Fig. 3 The orbit of Mexican Cap wavelet. (a) 2-D phase plane (g,g'); (b) 3-D phase space (g,g',ξ).

Taking

$$\eta=\frac{\xi}{\sqrt{2}}\,, \tag{13}$$

then equation (12) is reduced to

$$y''(\eta)+(5-\eta^2)y(\eta)=0\,, \tag{14}$$

which is the well-known first kind of Weber equation.

Similarly, let

$$x=\eta^2\,, \tag{15}$$

then equation (14) reduces to

$$y''(x) + \frac{2}{4x}y'(x) - \frac{1}{4}y(x) = 0\,. \tag{16}$$

From Eq. (16), we see that when $x \to \pm\infty$ (i.e. η and $\xi \to \pm\infty$), equation (16) is approximated to

$$y''(x) - \frac{1}{4}y(x) = 0\,. \tag{17}$$

From the physical view point, equation (17) is an ODE with negative restoring force. Obviously $(0,0)$ is a saddle point in phase plane (y, y'). Therefore, the orbit departs from unstable manifold in saddle point, then rotates around x near zero due to positive restoring force, at last comes back to stable manifold in saddle point due to negative restoring force, this forms a homoclinic orbit. The orbits in phase plane (g, g') and phase space (g, g', ξ) are shown in Figs. 3(a) and 3(b), respectively.

From Fig. 3, we see that when $\xi \to \pm\infty$, $g(\xi) \to 0$, then $g(\xi)$ is a solitary wave. It is impossible for linear ODE with constant coefficient.

5 The Other Closed Form Wavelet

Zayed and Walter[13−15] have constructed some wavelets in terms of elementary functions. These wavelets are called closed form wavelets, among which there is one written as

$$g(\xi) = -\frac{1}{\pi(\xi - 1/2)}\Big[\sin 2\pi\Big(\xi - \frac{1}{2}\Big) + \sin\pi\Big(\xi - \frac{1}{2}\Big) - 2\sin\frac{3\pi}{2}\Big(\xi - \frac{1}{2}\Big)\Big]\,. \tag{18}$$

Because it is constructed by linear combination of sinc function

$$\operatorname{sinc}\xi = g(\xi) = \frac{\sin\xi}{\xi}\,, \tag{19}$$

so we only consider $\operatorname{sinc}\xi$ function. It is easily shown that $\operatorname{sinc}\xi$ function (19) satisfies the following ODE with variable coefficient

$$g''(\xi) + \frac{2}{\xi}g'(\xi) + g(\xi) = 0\,, \tag{20}$$

or

$$\xi^2 g''(\xi) + 2\xi g'(\xi) + \xi^2 g(\xi) = 0\,, \tag{21}$$

which is zeroth-order spherical Bessel equation.

From Eq. (21) we see that when $\xi \to -\infty$, equation (21) denotes negative damping, while when $\xi \to +\infty$, positive damping. Therefore, $g(\xi) = 0$ is unstable focus when $\xi \to -\infty$, while $g(\xi) = 0$ is stable focus, when $\xi \to +\infty$. So $g(\xi) = 0$ is homoclinic point when $\xi \to \pm\infty$.

The orbit in 2-D phase plane (g, g') and 3-D phase space (g, g', ξ) are shown in Fig. 4(a) and Fig. 4(b), respectively. From Fig. 4, we see that sinc function (19) is a homoclinic orbit, while wavelet (18) is a solitary wave.

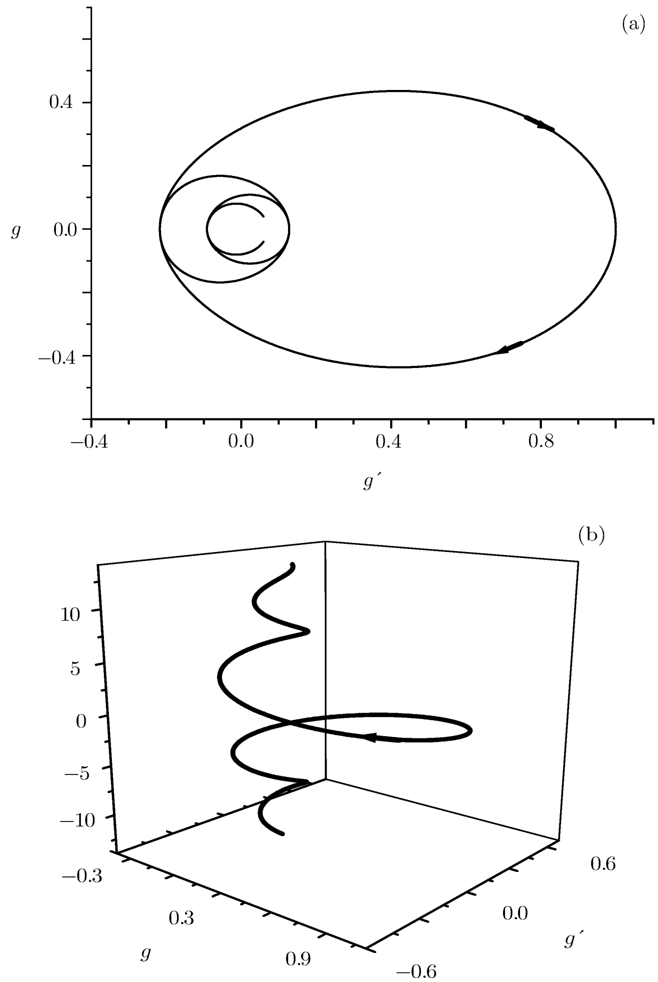

Fig. 4 The orbit of sinc function. (a) 2-D phase plane (g, g'); (b) 3-D phase space (g, g', ξ).

6 Conclusion

In this paper, it is shown that the homoclinic orbit exists in iterated functional system, so the solitary wave structure exists there, too. Moreover, Harr father wavelet, Mexican Cap wavelet, and other closed form wavelets all have this characteristic, so wavelet is certain kind of solitary wave.

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