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Structures of Equatorial Envelope Rossby Wave Under a Generalized External Forcing*

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Abstract *The cubic nonlinear Schrödinger (NLS for short) equation with a generalized external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow. And then various periodic structures for these equatorial envelope Rossby waves are obtained with the help of a new transformation, Jacobi elliptic functions, and elliptic equation. It is shown that different types of resonant phase-locked diabatic heating play different roles in structures of equatorial envelope Rossby wave.*

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1 Introduction

In the last decades, the theory of equatorial waves has attracted much more attention on equatorial atmospheric dynamics and nonlinear dynamics. It provides a dynamical frame to analyze the slowly evolving large-scale phenomena in low latitudes and underlining dynamics. These theories of equatorial waves have been used for various purposes, especially in explaining some fundamental features of tropical climate and global changes, such as Walker circulation,[1] the low-frequency Madden–Julian oscillation,[2] and ENSO.[3] Among the nonlinear theories for equatorial waves, many are related to nonlinear Rossby wave activity, for it can manifest some of the prime events of geophysical fluid flows, and this activity often leads to a large-scale localized coherent structure that has remarkable permanence and stability. When the zonal flow shear is taken to be nonuniform, one can derive Rossby solitary waves and envelope Rossby solitary waves. Benney,[4] Yamagata[5] and Zhao[6] investigated envelope Rossby solitary waves in barotropic shear and uniform or nonuniform flows, independently. However, none of them considered the effect of external sources, especially the influence of diabatic heating from oceans. In our last paper,[7] we applied the method of multi-scale expansion to derive the NLS equation with an external heating source satisfied by the large-amplitude equatorial Rossby waves. It reads

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=\eta Q_{11}(X,T) \tag{1}$$

with the following coordinate transformation defined by Jeffrey[8]

$$T=T_2\,,\qquad X=\frac{1}{\varepsilon}(X_2-c_gT_2)=X_1-c_gT_1\,, \tag{2}$$

where $Q_{11}(X,T)$ is the slowly varying external heating source, η denotes its strength, and ε is a small parameter.

In Ref. [7], we just considered two cases of diabatic heating. The first one is

$$Q_{11}(X,T)=0 \tag{3}$$

and then equation (1) reduces to the canonical NLS equation

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=0\,. \tag{4}$$

The second case is that the external heating source is an external phase-locked travelling wave source, i.e.

$$Q_{11}(X,T)=\mathrm{e}^{\mathrm{i}(kX-\omega T)}\,, \tag{5}$$

then equation (1) reduces to

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=\eta\,\mathrm{e}^{\mathrm{i}(kX-\omega T)}\,. \tag{6}$$

And there the basic structures of these two NLS equations without and with phase-locked source are obtained by using knowledge of Jacobi elliptic functions and elliptic equation. It is shown that phase-locked diabatic heating plays an important role in periodic structures of rational form.

If we suppose that equation (1) takes solution of the following form

$$A(X,T)=\phi(\xi)\exp[\mathrm{i}(kX-\omega T)]\,,\quad \xi=s(X-C_gT)\,, \tag{7}$$

and the external heating is chosen as

$$Q_{11}(X,T)=\psi(\xi)\,\mathrm{e}^{\mathrm{i}[kX-\omega T]}\,, \tag{8}$$

then equation (1) is rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=\frac{\gamma}{\alpha s^2}\phi-\frac{\delta}{\alpha s^2}\phi^3+\frac{\eta}{\alpha s^2}\psi \tag{9}$$

with

$$C_g=2\alpha k\,,\qquad -\gamma=\omega-\alpha k^2\,. \tag{10}$$

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The two cases of external heating considered in Ref. [7] are

$$\psi(\xi) = 0\,, \tag{11}$$

and

$$\psi(\xi) = 1\,. \tag{12}$$

There are still more types of $\psi(\xi)$, in Ref. [9] we considered the quadratic external heating, i.e.

$$\psi(\xi) = \phi^2\,, \tag{13}$$

which can be taken as a resonant forcing. There more new type structures, which are different from those obtained in Ref. [7], were derived. So we can say that different external forcing will lead to different structures of equatorial envelope Rossby waves.

In this paper, we will introduce a generalized external heating to further discuss the influence of different external heating on the structures of the equatorial envelope Rossby wave.

2 Structures to NLS Equation with a Generalized External Heating Source

The generalized external forcing is taken as a finite expansion of ϕ and the highest degree of ϕ is no more than the degree in Eq. (9), i.e.

$$\psi(\xi) = d_0 + d_1\phi + d_2\phi^2 + d_3\phi^3\,, \tag{14}$$

then equation (9) is rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = b_0 + b_1\phi + b_2\phi^2 + b_3\phi^3 \tag{15}$$

with

$$b_0 = \frac{\eta d_0}{\alpha s^2}\,, \qquad b_1 = \frac{\gamma + \eta d_1}{\alpha s^2}\,,$$
$$b_2 = \frac{\eta d_2}{\alpha s^2}\,, \qquad b_3 = \frac{\eta d_3 - \delta}{\alpha s^2}\,. \tag{16}$$

Obviously, $d_0 = d_1 = d_2 = d_3 = 0$ and $d_1 = d_2 = d_3 = 0$, $d_0 \neq 0$ are the two cases we considered in Ref. [7], while $d_0 = d_1 = d_3 = 0$, $d_2 \neq 0$ is the case we considered in Ref. [9]. Actually, $d_2 = d_3 = 0$, $d_0 \neq 0$, $d_1 \neq 0$, and $d_0 = d_2 = d_3 = 0$, $d_1 \neq 0$ are similar to the two cases we considered in Ref. [7], while $d_0 = d_3 = 0$, $d_1 \neq 0$, $d_2 \neq 0$ and $d_0 = 0$, $d_1 \neq 0$, $d_2 \neq 0$, $d_3 \neq 0$ are similar to the case we considered in Ref. [9]. So in the following we consider the case with $d_0 \neq 0$.

Equation (15) is just a kind of elliptic equation, which cannot be easily solved directly. In order to solve equation of this form, certain transformation must be introduced, just like Legendre's transformation. Here the transformation we introduce is of the following form,

$$\phi(\xi) = \frac{1}{c_0 + c_1\varphi^2}\,, \qquad c_1 \neq 0\,, \tag{17}$$

then

$$\phi' = -\frac{2c_1\varphi\varphi'}{(c_0 + c_1\varphi^2)^2}\,, \tag{18}$$

$$\phi'' = \frac{8c_1^2\varphi^2\varphi'^2 - 2c_1(c_0 + c_1\varphi^2)(\varphi\varphi'' + \varphi'^2)}{(c_0 + c_1\varphi^2)^3}\,. \tag{19}$$

In order to reach reasonable results, φ must satisfy

$$\varphi'^2 = a + b\varphi^2 + c\varphi^4\,, \tag{20}$$

i.e.

$$\varphi'' = b\varphi + 2c\varphi^3\,. \tag{21}$$

Combining Eqs. (17), (20), and (21) with Eq. (15) yields

$$\begin{aligned}
&-2ac_0c_1 = b_0c_0^3 + b_1c_0^2 + b_2c_0 + b_3\,,\\
&6ac_1^2 - 4bc_0c_1 = 3b_0c_0^2c_1 + 2b_1c_0c_1 + b_2c_1\,,\\
&4bc_1^2 - 6cc_0c_1 = 3b_0c_0c_1^2 + b_1c_1^2\,,\\
&2cc_1^2 = b_0c_1^3\,,
\end{aligned} \tag{22}$$

from which we have

$$c_0 = \frac{4b - b_1}{6b_0}\,, \qquad c_1 = \frac{2c}{6b_0} \tag{23}$$

with constraints

$$144ac = (4b - b_1)^2 + 4(2b + b_1)(4b - b_1) + 12b_0b_2\,, \tag{24}$$

$$144ac(4b - b_1) + (4b - b_1)^3 + 6b_1(4b - b_1)^2 + 36b_0b_2(4b - b_1) + 216b_0^2b_3 = 0\,. \tag{25}$$

Equation (20) or equation (21) has many more kinds of solutions, we will show some next expressed in terms of different Jacobi elliptic functions.[10]

(i) If $a = 1$, $b = -(1 + m^2)$, and $c = m^2$, then the solution is

$$\varphi_1 = \mathrm{sn}(\xi, m)\,, \tag{26}$$

$$\phi_1 = \frac{6b_0}{b_m + 12m^2\mathrm{sn}^2(\xi, m)}\,, \quad b_m = -4(1 + m^2) - b_1 \tag{27}$$

with the constraints

$$3b_m^2 + 24(1 + m^2)b_m - 12b_0b_2 + 144m^2 = 0\,, \tag{28}$$

$$b_m^3 + 6b_1b_m^2 + (36b_0b_2 + 144m^2)b_m + 216b_0^2b_3 = 0\,, \tag{29}$$

where $0 \leq m \leq 1$, and is called modulus of Jacobi elliptic functions,[10−14] and $\mathrm{sn}(\xi, m)$ is Jacobi elliptic sine function.[10−14]

(ii) If $a = 1 - m^2$, $b = 2m^2 - 1$, and $c = -m^2$, then the solution is

$$\varphi_2 = \mathrm{cn}(\xi, m)\,, \tag{30}$$

$$\phi_2 = \frac{6b_0}{b_m - 12m^2\mathrm{cn}^2(\xi, m)}\,, \quad b_m = 4(2m^2 - 1) - b_1 \tag{31}$$

with the constraints

$$144m^2(1 - m^2) - 3b_m^2 + 24(2m^2 - 1)b_m + 12b_0b_2 = 0\,, \tag{32}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 - 144m^2(1 - m^2)]b_m + 216b_0^2b_3 = 0, \tag{33}$$

where $\mathrm{cn}(\xi, m)$ is Jacobi elliptic cosine function.[10−14]

(iii) If $a = 1 - m^2$, $b = 2 - m^2$, and $c = -1$, then the solution is

$$\varphi_3 = \mathrm{dn}(\xi, m)\,, \tag{34}$$

$$\phi_3 = \frac{6b_0}{b_m - 12\,\mathrm{dn}^2(\xi,m)}\,, \qquad b_m = 4(2-m^2) - b_1 \tag{35}$$

with the constraints

$$144(1-m^2) - 3b_m^2 + 24(2-m^2)b_m + 12b_0b_2 = 0\,, \tag{36}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 - 144(1-m^2)]b_m + 216b_0^2b_3 = 0\,, \tag{37}$$

where $\mathrm{dn}(\xi,m)$ is Jacobi elliptic function of the third kind.[10−14]

(iv) If $a = m^2$, $b = -(1+m^2)$, and $c = 1$, then the solution is

$$\varphi_4 = \mathrm{ns}(\xi,m) \equiv \frac{1}{\mathrm{sn}(\xi,m)}\,, \tag{38}$$

$$\phi_4 = \frac{6b_0}{b_m + 12\mathrm{ns}^2(\xi,m)}\,, \quad b_m = -4(1+m^2) - b_1\,, \tag{39}$$

with the constraints

$$144m^2 = -3b_m^2 - 24(1+m^2)b_m + 12b_0b_2\,, \tag{40}$$

$$b_m^3 + 6b_1b_m^2 + (36b_0b_2 + 144m^2)b_m + 216b_0^2b_3 = 0\,. \tag{41}$$

(v) If $a = -m^2$, $b = 2m^2 - 1$, and $c = 1 - m^2$, then the solution is

$$\varphi_5 = \mathrm{nc}(\xi,m) \equiv \frac{1}{\mathrm{cn}(\xi,m)}\,, \tag{42}$$

$$\phi_5 = \frac{6b_0}{b_m + 12(1-m^2)\mathrm{nc}^2(\xi,m)}\,,$$

$$b_m = 4(2m^2-1) - b_1 \tag{43}$$

with the constraints

$$144m^2(1-m^2) - 3b_m^2 + 24(2m^2-1)b_m + 12b_0b_2 = 0\,, \tag{44}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 - 144m^2(1-m^2)]b_m + 216b_0^2b_3 = 0\,. \tag{45}$$

(vi) If $a = -1$, $b = 2 - m^2$, and $c = m^2 - 1$, then the solution is

$$\varphi_6 = \mathrm{nd}(\xi,m) \equiv \frac{1}{\mathrm{dn}(\xi,m)}\,, \tag{46}$$

$$\phi_6 = \frac{6b_0}{b_m + 12(m^2-1)\mathrm{nd}^2(\xi,m)}\,,$$

$$b_m = 4(2-m^2) - b_1 \tag{47}$$

with the constraints

$$144(m^2-1) - 3b_m^2 + 24(2-m^2)b_m + 12b_0b_2 = 0\,, \tag{48}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 - 144(m^2-1)]b_m + 216b_0^2b_3 = 0\,. \tag{49}$$

(vii) If $a = 1$, $b = 2 - m^2$, and $c = 1 - m^2$, then the solution is

$$\varphi_7 = \mathrm{sc}(\xi,m) \equiv \frac{\mathrm{sn}(\xi,m)}{\mathrm{cn}(\xi,m)}\,, \tag{50}$$

$$\phi_7 = \frac{6b_0}{b_m + 12(1-m^2)\mathrm{sc}^2(\xi,m)}\,, \quad b_m = 4(2-m^2) - b_1 \tag{51}$$

with the constraints

$$144(1-m^2) = -3b_m^2 + 24(2-m^2)b_m + 12b_0b_2\,, \tag{52}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 + 144(1-m^2)]b_m + 216b_0^2b_3 = 0\,. \tag{53}$$

(viii) If $a = 1$, $b = 2m^2 - 1$, and $c = (m^2-1)m^2$, then the solution is

$$\varphi_8 = \mathrm{sd}(\xi,m) \equiv \frac{\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)}\,, \tag{54}$$

$$\phi_8 = \frac{6b_0}{b_m + 12(m^2-1)m^2\mathrm{sd}^2(\xi,m)}\,,$$

$$b_m = 4(2m^2-1) - b_1 \tag{55}$$

with the constraints

$$144(m^2-1)m^2 = -3b_m^2 + 24(2m^2-1)b_m + 12b_0b_2\,, \tag{56}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 + 144(m^2-1)m^2]b_m + 216b_0^2b_3 = 0\,. \tag{57}$$

(ix) If $a = 1 - m^2$, $b = 2 - m^2$, and $c = 1$, then the solution is

$$\varphi_9 = \mathrm{cs}(\xi,m) \equiv \frac{\mathrm{cn}(\xi,m)}{\mathrm{sn}(\xi,m)}\,, \tag{58}$$

$$\phi_9 = \frac{6b_0}{b_m + 12\mathrm{cs}^2(\xi,m)}\,, \quad b_m = 4(2-m^2) - b_1 \tag{59}$$

with the constraints

$$144(1-m^2) = -3b_m^2 + 24(2-m^2)b_m + 12b_0b_2\,, \tag{60}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 + 144(1-m^2)]b_m + 216b_0^2b_3 = 0\,. \tag{61}$$

(x) If $a = 1$, $b = -(1+m^2)$, and $c = m^2$, then the solution is

$$\varphi_{10} = \mathrm{cd}(\xi,m) \equiv \frac{\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}\,, \tag{62}$$

$$\phi_{10} = \frac{6b_0}{b_m + 12m^2\mathrm{cd}^2(\xi,m)}\,, \quad b_m = -4(1+m^2) - b_1 \tag{63}$$

with the constraints

$$144m^2 = -3b_m^2 - 24(1+m^2)b_m + 12b_0b_2\,, \tag{64}$$

$$b_m^3 + 6b_1b_m^2 + (36b_0b_2 + 144m^2)b_m + 216b_0^2b_3 = 0\,. \tag{65}$$

(xi) If $a = m^2(m^2-1)$, $b = 2m^2 - 1$, and $c = 1$, then the solution is

$$\varphi_{11} = \mathrm{ds}(\xi,m) \equiv \frac{\mathrm{dn}(\xi,m)}{\mathrm{sn}(\xi,m)}\,, \tag{66}$$

$$\phi_{11} = \frac{6b_0}{b_m + 12\mathrm{ds}^2(\xi,m)}\,, \quad b_m = 4(2m^2-1) - b_1 \tag{67}$$

with the constraints

$$144m^2(m^2-1) = -3b_m^2 + 24(2m^2-1)b_m + 12b_0b_2\,, \tag{68}$$

$$b_m^3 + 6b_1b_m^2 + [36b_0b_2 + 144m^2(m^2-1)]b_m + 216b_0^2b_3 = 0\,. \tag{69}$$

(xii) If $a = m^2$, $b = -(1+m^2)$, and $c = 1$, then the solution is

$$\varphi_{12} = \mathrm{dc}(\xi,m) \equiv \frac{\mathrm{dn}(\xi,m)}{\mathrm{cn}(\xi,m)}\,, \tag{70}$$

$$\phi_{12} = \frac{6b_0}{b_m + 12\mathrm{dc}^2(\xi,m)}\,, \quad b_m = -4(1+m^2) - b_1 \tag{71}$$

with the constraints

$$144m^2 = -3b_m^2 - 24(1+m^2)b_m + 12b_0b_2\,, \tag{72}$$

$$b_m^3 + 6b_1b_m^2 + (36b_0b_2 + 144m^2)b_m + 216b_0^2b_3 = 0\,. \tag{73}$$

There still exist many other kinds of solutions in terms of Jacobi elliptic functions,[15−17] which we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions.

3 Conclusion and Discussion

A simple shallow-water model with influence of diabatic heating on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the asymptotic method of multiple scales, the cubic nonlinear Schrödinger equation with an external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow.[7] And then various periodic structures for these equatorial envelope Rossby waves are obtained with the help of Jacobi elliptic functions and elliptic equation. It is shown that the results are different for equatorial envelope Rossby waves without a source and with a phase-locked diabatic heating source. They have different structures due to the phase-locked diabatic heating source, and the phase-locked diabatic heating source plays an important role in forming periodic structures of rational form. Of course, these periodic structures contain solitons, solitary waves, and singular structures, and they also have their different practical applications in explaining atmospheric events. Moreover, in Ref. [9], we considered another special case of external heating and found some new exact results.

Here we can see the external forcing plays an important role in two aspects. The first one is the basic state d_0 or b_0, which is in proportion to the external strength. This results in different structures for equatorial envelope Rossby wave, obviously the strength of equatorial envelope Rossby wave depend on d_0 or b_0. And the second one is the modulation of b_1, b_2, and b_3, which also leads to different structures for equatorial envelope Rossby wave. Moreover, the external heating results in structures for equatorial envelope Rossby wave of rational form, for example, from ϕ_1 to ϕ_{12}. Different from the results obtained in Ref. [7], here the solutions of rational form depends on the constraints composed of b_0, b_1, b_2, b_3. So we can say that different types of external heating will lead to different structures for equatorial envelope Rossby wave.

It needs more further research for more various heating sources, for this effort provides a better understanding of function from external heating sources and their impacts on the equatorial Rossby waves and climate changes.

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