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Solitary Wave and Wave Front as Viewed From Curvature*

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Abstract *The solitary wave and wave front are two important behaviors of nonlinear evolution equations. Geometrically, solitary wave and wave front are all plane curve. In this paper, they can be represented in terms of curvature $c(s)$, which varies with arc length s. For solitary wave when $s \longrightarrow \pm\infty$, then its curvature $c(s)$ approaches zero, and when $s = 0$, the curvature $c(s)$ reaches its maximum. For wave front, when $s \longrightarrow \pm\infty$, then its curvature $c(s)$ approaches zero, and when $s = 0$, the curvature $c(s)$ is still zero, but $c'(s) \neq 0$. That is, $s = 0$ is a turning point. When $c(s)$ is given, the variance at some point (x, y) in stream line with arc length s satisfies a 2-order linear variable-coefficient ordinary differential equation. From this equation, it can be determined qualitatively whether the given curvature is a solitary wave or wave front.*

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Key words: curvature, solitary waves, wave front, homoclinic orbit

1 Introduction

Solitary wave and wave front are two important behaviors of nonlinear evolution equations,[1−8] whose analytical solutions have been obtained from many methods.[9−14] However, it is still very difficult to find some analytical representations for solitary wave and wave front. Qualitatively, it has been shown that the solitary wave and wave front exist in conservative or dissipative partial differential equations, where the solitary wave corresponds to the homoclinic orbit for ordinary differential equations, and wave front to the heteroclinic orbit.[8] Because the equilibrium states to connect homoclinic (or heteroclinic) orbit include not only saddle point, focus point, node point, saddle-focus point, but also limit cycle, thus the curve of homoclinic (or heteroclinic) orbit is very complicated. In this paper, starting from the geometry,[15,16] the connection of solitary wave (or wave front) with curvature $c(s)$ of plane curve is established and then is applied to understand the essence of solitary wave and wave front.

2 Curvature and Curvature Characteristics of Solitary Wave and Wave Front

Mathematically, the curvature c of a curve in a plane is defined as derivative of tangent direction with respect to arc length s

$$c \equiv \frac{\mathrm{d}\alpha}{\mathrm{d}s}\,, \tag{1}$$

where α is the angle between the tangent of the curve and x axis, and s is the arc length. Generally, if the curve is counter-clockwise rotation, then α increases with s, and $c > 0$. If curve is clockwise rotation, then α decreases with s, and $c < 0$.

The reciprocal of curvature c is called curvature radius. The variation of curvature implies the variance of crooked degree for a curve. When the curve is very straight, the curvature radius is very large and the curvature is small. The smaller the curvature radius, the larger the curvature, and the larger the crooked degree of the curve.

The curve shape of solitary wave and wave front is shown as Fig. 1.

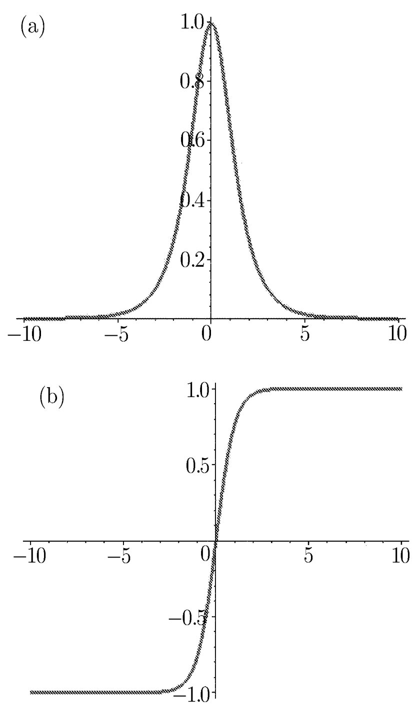

Fig. 1 The typical curve shape. (a) solitary wave; (b) wave front.

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Physically, when we consider the travelling wave solution of nonlinear evolution, the abscissa of Fig. 1 denotes wave variable $\xi = x - ct$, where c is wave speed. When we consider homoclinic (or heteroclinic) orbit, the abscissa of Fig. 1 denotes time t. Hence for solitary wave when ξ or t approaches $\pm\infty$, the value of solitary wave approaches the same value, in Fig. 1(a) $u \to 0$, but for wave front, it approaches different values, in Fig. 1(b) $u \to 1$ and $u \to 0$.

Now from the view of curvature, if the starting point $s = 0$ of arc length is taken, then the arc length on the right-hand side of $s = 0$ is positive, while the value on the left side is negative. Thus for solitary wave, we have

$$\begin{aligned} &c(s) \to 0, \quad \text{as} \quad s \to \pm\infty\,; \\ &c(s) \to \text{maximum}, \quad \text{as} \quad s \to 0\,. \end{aligned} \tag{2}$$

For wave front, we have

$$\begin{aligned} &c(s) \to 0, \quad \text{as} \quad s \to \pm\infty\,; \\ &c(s) = 0, \quad c'(s) \neq 0, \quad \text{as} \quad s \to 0\,. \end{aligned} \tag{3}$$

For example, there are the following two curvatures

$$c(s) = \mathrm{e}^{-s^2/2}\,, \tag{4}$$

and

$$c(s) = -s\,\mathrm{e}^{-s^2/2}\,. \tag{5}$$

For curvature (4), obviously, when $s \to \pm\infty$, $c(s) \to 0$ and $c(0) = 1$ is a maximum. It is consistent with curvature condition (2) of solitary wave. For curvature (5), when $s \to \pm\infty$, $c(s) \to 0$ and $c'(s) = -1 \neq 0$, it is consistent with curvature condition (3).

3 Differential Equation of $\mathrm{d}\boldsymbol{x}/\mathrm{d}\boldsymbol{s}$ and $\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{s}$

Though the above two curvatures (4) and (5) are very simple, they are consistent with curvature condition (2) or (3) of solitary wave or wave front. But, are they solitary wave or wave front from physics? So, we will use the variance of point (x, y) in curve with arc length s,

$$p = \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\alpha, \quad q = \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\alpha\,. \tag{6}$$

The first and second derivative of Eq. (6) with respect to arc length s yields

$$\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{\mathrm{d}^2x}{\mathrm{d}s^2} = \sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s} = -c(s)q\,, \\ &\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\mathrm{d}^2y}{\mathrm{d}s^2} = \cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s} = c(s)p\,, \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{\mathrm{d}^2p}{\mathrm{d}s^2} = -c'(s)q - c(s)\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{c'(s)}{c(s)}\frac{\mathrm{d}p}{\mathrm{d}s} - c^2(s)p\,, \\ &\frac{\mathrm{d}^2q}{\mathrm{d}s^2} = c'(s)p - c(s)\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{c'(s)}{c(s)}\frac{\mathrm{d}q}{\mathrm{d}s} - c^2(s)q\,. \end{aligned} \tag{8}$$

Equation (8) shows that $p(s)$ and $q(s)$ satisfy the following 2-order ordinary differential equation with variable coefficients,

$$\frac{\mathrm{d}^2z}{\mathrm{d}s^2} + a(s)\frac{\mathrm{d}z}{\mathrm{d}s} + b(s)z = 0\,, \tag{9}$$

where

$$a(s) = -\frac{c'(s)}{c(s)}, \quad b(s) = c^2(s)\,. \tag{10}$$

The physical meaning of equation Eq. (9) is that the three terms on the left side denote acceleration, damping force, and restoring force,[4,5,17,18] respectively. $a(s)$ and $b(s)$ in equation (10) denote damping coefficient and restoring force coefficient.

In the following, we can analyze qualitatively the equation (9) and determine whether curvatures (4) and (5) are solitary wave or wave front from physical view.

4 Qualitative Analysis of Solitary Wave and Wave Front

Substituting curvature (4) into Eqs. (9) and (10) yields

$$z'' + sz' + (\mathrm{e}^{-s^2/2})^2 z = 0\,. \tag{11}$$

Let

$$z = \mathrm{e}^{-s^2/2}u\,, \tag{12}$$

then equation (11) reduces to

$$u'' + \Big(\frac{1}{2} - \frac{s^2}{4} + \mathrm{e}^{-s^2}\Big)u = 0\,. \tag{13}$$

Setting $u' = v$, then equation (13) reduces to

$$u' = v, \quad v' = \Big(\frac{1}{2} - \frac{s^2}{4} + \mathrm{e}^{-s^2}\Big)u\,. \tag{14}$$

Equation (14) shows that when $s \to \pm\infty$, the steady solution $u = 0$, $v = 0$ of Eq. (14) is controlled by the negative restoring force. Hence, the steady solution is a saddle point, as shown in Fig. 2(a).

Therefore, the homoclinic orbit, which starts from unstable manifold of saddle point and comes back to stable manifold of saddle point, is formed. This is a solitary wave.

Similarly, substituting curvature (5) into Eqs. (9) and (10) yields

$$z'' - \Big(\frac{1}{s} - s\Big)z' + (-s\,\mathrm{e}^{-s^2/2})^2 z = 0\,. \tag{15}$$

Equation (15) shows that when $s \to -\infty$, the steady solution of Eq. (15) is controlled by negative damping force and positive restoring force, but damping force is larger, hence the steady solution is an unstable node. When $s \to +\infty$, the steady solution of Eq. (15) is controlled by positive damping force and positive restoring force. Because the damping force is larger, hence the steady solution is a stable node as shown in Fig. 2(b). So, starting from unstable node of $s \to -\infty$ and coming back to stable node of $s \to +\infty$ form a heteroclinic orbit. This is a wave front.

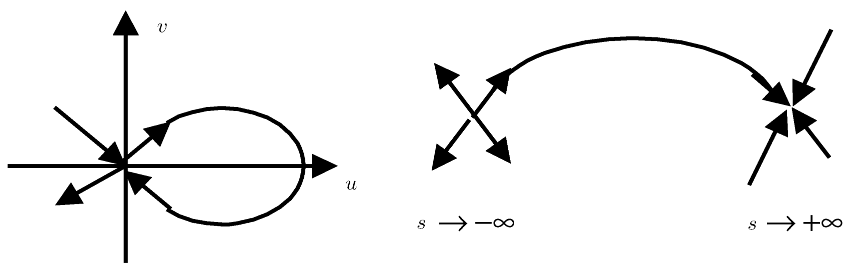

Fig. 2 Schematic steady solution of $s \to \pm\infty$. (a) homoclinic orbit; (b) heteroclinic orbit.

5 Conclusion

The solitary wave and wave front are two important behaviors of nonlinear evolution equations. Geometrically, solitary wave and wave front are all plane curve. In this paper, they have been represented in terms of curvature $c(s)$, which varies with arc length s. If the arc length is calculated from its starting point and forwards, then the arc length is positive; if it is calculated from its starting point and backwards, then the arc length is negative. Hence, for solitary wave when $s \longrightarrow \pm\infty$, then its curvature $c(s)$ approaches zero, and when $s = 0$, the curvature $c(s)$ reaches its maximum. For wave front, when $s \longrightarrow \pm\infty$, then its curvature $c(s)$ approaches zero, and when $s = 0$, the curvature $c(s)$ is still zero, but $c'(s) \neq 0$. This is, $s = 0$ is a turning point. When $c(s)$ is given, the variance at some point (x, y) in stream line with arc length s satisfies 2-order linear variable-coefficient ordinary differential equation. From this equation, it can be determined qualitatively whether the given curvature is a solitary wave or wave front.

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