Available online at www.sciencedirect.com
SCIENCE DIRECT
Physics Letters A 336 (2005) 175–179
PHYSICS LETTERS A
www.elsevier.com/locate/pla

# Periodic solutions for a class of coupled nonlinear partial differential equations

Shikuo Liu [a], Zuntao Fu [a,b,*], Shida Liu [a,b]

[a] *School of Physics, Peking University, Beijing 100871, China* [1]
[b] *LTCS, Peking University, Beijing 100871, China*

Received 16 June 2004; accepted 12 January 2005

Communicated by A.R. Bishop

**Abstract**

In this Letter, by applying the Jacobi elliptic function expansion method, the periodic solutions for three coupled nonlinear partial differential equations are obtained.


*PACS:* 03.65.Ge

*Keywords:* Jacobi elliptic function; Periodic wave solution; Nonlinear partial differential equation

## 1. Introduction

Recently, Hu presented a new method finding exact traveling wave solutions of coupled nonlinear differential equations [1,2]. This new ansatz method, in which a simple rational polynomial relation is assumed to exist between dependent variables in the coupled differential equations, was successfully applied to obtain some new solutions to three kinds of coupled differential equations of mathematical physics. However, there only some soliton-like solutions were derived and some conditions are coarse. In this Letter, by using the Jacobi elliptic function expansion method [3–5], we obtain the periodic solutions for a class of coupled nonlinear partial differential equations, which play an important role in modern physics.

* Corresponding author.
*E-mail address:* fuzt@pku.edu.cn (Z. Fu).
[1] Correspondence address.

0375-9601/$ – see front matter 
doi:10.1016/j.physleta.2005.01.025

## 2. Periodic solutions for coupled nonlinear plasma system

The coupled nonlinear plasma system [6,7] reads

$$u_{xx} = \alpha_1 u + \alpha_2 uv, \tag{1a}$$

$$v_{xx} = \beta_1 v + \beta_2 v^2 + \beta_3 u^2, \tag{1b}$$

when $\beta_3 = 0$, this implies that $v$ is independent on $u$. Hu [1,2] obtained some new soliton-like solutions to Eqs. (1).

By using the Jacobi elliptic function expansion method [3–5], $u$ and $v$ can be expressed as

$$u = a_0 + a_1 \operatorname{sn}\xi + a_2 \operatorname{sn}^2\xi, \tag{2a}$$

$$v = b_0 + b_1 \operatorname{sn}\xi + b_2 \operatorname{sn}^2\xi, \tag{2b}$$

where $\xi = kx$, $\operatorname{sn}\xi$ is the Jacobi elliptic sine function [8–11].

Substituting Eqs. (2) into Eqs. (1) leads to a set of algebraic equations for $\operatorname{sn}^i\xi$ $(i = 0, 1, 2, 3, 4)$, from which one has

$$\begin{aligned}
&a_1 = b_1 = 0, \qquad b_2 = \frac{6m^2k^2}{\alpha_2}, \qquad a_2 = \pm\sqrt{\frac{\alpha_2 - \beta_2}{\beta_3}}\, b_2, \\
&a_0 = \mp\sqrt{\frac{\alpha_2 - \beta_2}{\beta_3}}\left[\frac{2(1+m^2)k^2}{\alpha_2} + \frac{\alpha_2\beta_1 - 2\alpha_1\beta_2}{2\alpha_2(\alpha_2 - 2\beta_2)}\right], \\
&b_0 = -\frac{2(1+m^2)k^2}{\alpha_2} - \frac{2\alpha_1\alpha_2 - 2\alpha_1\beta_2 - \alpha_2\beta_1}{2\alpha_2(\alpha_2 - 2\beta_2)}, \\
&k^2 = \frac{(\alpha_1 + \alpha_2 b_0)a_0}{2a_2} = \frac{(\beta_1 + \beta_2 b_0)b_0 + \beta_3 a_0^2}{2b_2},
\end{aligned} \tag{3}$$

with $m$ $(0 < m < 1)$ is the modulus.

So, the periodic solutions to coupled nonlinear plasma system (1) are

$$u = a_0 + a_2 \operatorname{sn}^2\xi = (a_0 + a_2) - a_2 \operatorname{cn}^2\xi = a_0 + \frac{a_2}{m^2} - \frac{a_2}{m^2}\operatorname{dn}^2\xi, \tag{4a}$$

$$v = b_0 + b_2 \operatorname{sn}^2\xi = (b_0 + b_2) - b_2 \operatorname{cn}^2\xi = b_0 + \frac{b_2}{m^2} - \frac{b_2}{m^2}\operatorname{dn}^2\xi, \tag{4b}$$

where $\operatorname{cn}\xi$ and $\operatorname{dn}\xi$ are the Jacobi elliptic cosine function and Jacobi elliptic function of the third kind [8–11].

When $m \to 1$, Eqs. (4) reduce to the following solitary wave solutions

$$u = (a_0 + a_2) - a_2 \operatorname{sech}^2\xi, \qquad v = (b_0 + b_2) - b_2 \operatorname{sech}^2\xi. \tag{5}$$

## 3. Periodic solutions for coupled physical system

The coupled physical system [12,13] reads

$$u_{xx} = \alpha_1 u + \alpha_2 u^3 + \alpha_3 uv^2, \tag{6a}$$

$$v_{xx} = \beta_1 v + \beta_2 v^3 + \beta_3 v\left(u^2 - 1\right). \tag{6b}$$

Similarly, we seek the periodic solutions of Eqs. (6) in the form

$$u = a_0 + a_1 \operatorname{sn}\xi, \qquad v = b_0 + b_1 \operatorname{sn}\xi, \quad \xi = kx. \tag{7}$$

Substituting (7) into Eqs. (6) leads to the following results

$$a_0^2=\frac{\alpha_3(\beta_1-\beta_3)-\alpha_1\beta_2}{\alpha_2\beta_2-\alpha_3\beta_3},\qquad b_0^2=\frac{\alpha_1\beta_3-\alpha_2(\beta_1-\beta_3)}{\alpha_2\beta_2-\alpha_3\beta_3},$$
$$b_1^2=\frac{2k^2m^2(\alpha_2-\beta_3)}{\alpha_2\beta_2-\alpha_3\beta_3},\qquad a_1^2=\frac{2k^2m^2(\beta_2-\alpha_3)}{\alpha_2\beta_2-\alpha_3\beta_3},$$
$$k^2=-\frac{(\alpha_1+3\alpha_2a_0^2+\alpha_3b_0^2)a_1+2\alpha_3a_0b_0b_1}{(1+m^2)a_1}=-\frac{(\beta_1-\beta_3+3\beta_2b_0+\beta_3a_0^2)b_1+2\beta_3a_0b_0a_1}{(1+m^2)b_1}. \tag{8}$$

When $m\to 1$, (7) reduces to

$$u=a_0+a_1\tanh\xi,\qquad v=b_0+b_1\tanh\xi,\quad \xi=kx. \tag{9}$$

Similar to (7), we have

$$u=a_0+a_1\operatorname{cn}\xi,\qquad v=b_0+b_1\operatorname{cn}\xi,\quad \xi=kx, \tag{10}$$

with $a_0$ and $b_0$ same as (8), but

$$b_1^2=-\frac{2k^2m^2(\alpha_2-\beta_3)}{\alpha_2\beta_2-\alpha_3\beta_3},\qquad a_1^2=-\frac{2k^2m^2(\beta_2-\alpha_3)}{\alpha_2\beta_2-\alpha_3\beta_3},$$
$$k^2=\frac{(\alpha_1+3\alpha_2a_0^2+\alpha_3b_0^2)a_1+2\alpha_3a_0b_0b_1}{(2m^2-1)a_1}=\frac{(\beta_1-\beta_3+3\beta_2b_0+\beta_3a_0^2)b_1+2\beta_3a_0b_0a_1}{(2m^2-1)b_1}, \tag{11}$$

and

$$u=a_0+a_1\operatorname{dn}\xi,\qquad v=b_0+b_1\operatorname{dn}\xi,\quad \xi=kx, \tag{12}$$

with $a_0$ and $b_0$ same as (8), but

$$b_1^2=-\frac{2k^2(\alpha_2-\beta_3)}{\alpha_2\beta_2-\alpha_3\beta_3},\qquad a_1^2=-\frac{2k^2(\beta_2-\alpha_3)}{\alpha_2\beta_2-\alpha_3\beta_3},$$
$$k^2=\frac{(\alpha_1+3\alpha_2a_0^2+\alpha_3b_0^2)a_1+2\alpha_3a_0b_0b_1}{(2-m^2)a_1}=\frac{(\beta_1-\beta_3+3\beta_2b_0+\beta_3a_0^2)b_1+2\beta_3a_0b_0a_1}{(2-m^2)b_1}. \tag{13}$$

When $m\to 1$, (10) and (12) reduce to

$$u=a_0+a_1\operatorname{sech}\xi,\qquad v=b_0+b_1\operatorname{sech}\xi,\quad \xi=kx. \tag{14}$$

## 4. Periodic solutions for generalized DS equations

The generalized Drinfelf–Sokolov (DS for short) equations [14] can be written as

$$u_t+\alpha_1uu_x+\beta_1u_{xxx}+\gamma\left(v^\delta\right)_x=0, \tag{15a}$$
$$v_t+\alpha_2uv_x+\beta_2v_{xxx}=0. \tag{15b}$$

We seek the traveling wave solutions of Eqs. (15) in the form

$$u=u(\xi),\qquad v=v(\xi),\quad \xi=k(x-ct), \tag{16}$$

where $k$ and $c$ are wave number and wave speed, respectively.

Substituting (16) into (15), we have

$$-c\frac{du}{d\xi}+\alpha_1 u\frac{du}{d\xi}+\beta_1 k^2\frac{d^3u}{d\xi^3}+\gamma\frac{d}{d\xi}\left(v^\delta\right)=0, \tag{17a}$$

$$-c\frac{dv}{d\xi}+\alpha_2 u\frac{dv}{d\xi}+\beta_2 k^2\frac{d^3v}{d\xi^3}=0, \tag{17b}$$

In order to solve Eqs. (17), the following transformation

$$w=v^{\delta/2}, \tag{18}$$

is introduced, then Eqs. (17) can be rewritten as

$$-c\frac{du}{d\xi}+\alpha_1 u\frac{du}{d\xi}+\beta_1 k^2\frac{d^3u}{d\xi^3}+2\gamma w\frac{dw}{d\xi}=0, \tag{19a}$$

$$-cw^2\frac{dw}{d\xi}+\alpha_2 uw^2\frac{dw}{d\xi}+\beta_2 k^2\left[\left(\frac{2}{\delta}-1\right)\left(\frac{2}{\delta}-2\right)\left(\frac{dw}{d\xi}\right)^3+3\left(\frac{2}{\delta}-1\right)w\frac{dw}{d\xi}\frac{d^2w}{d\xi^2}+w^2\frac{d^3w}{d\xi^3}\right]=0. \tag{19b}$$

Similarly, the formal solutions can be written as

$$u=a_0+a_1\operatorname{sn}\xi+a_2\operatorname{sn}^2\xi, \tag{20a}$$

$$v=b_0+b_1\operatorname{sn}\xi+b_2\operatorname{sn}^2\xi. \tag{20b}$$

Substituting (20) into (19), we have

$$a_1=b_1=0, \qquad a_2=-\frac{2(4+\delta)(2+\delta)m^2\beta_2 k^2}{\alpha_2\delta^2},$$
$$b_2=\pm\frac{2m^2k^2}{\alpha_2\delta^2}\sqrt{\frac{\beta_2(4+\delta)(2+\delta)}{\gamma}\left[6\delta^2\alpha_2\beta_1-(4+\delta)(2+\delta)\alpha_1\beta_2\right]}, \tag{21}$$

and

$$\left[-c+\alpha_2 a_0-4\left(1+m^2\right)\beta_2 k^2\right]b_0+6\left(\frac{2}{\delta}-1\right)\beta_2 k^2 b_2=0,$$
$$2\left[\alpha_2 a_2+3m^2\left(\frac{6}{\delta}+1\right)\beta_2 k^2\right]b_0+\left[-c+\alpha_2 a_0-\frac{16}{\delta^2}\left(1+m^2\right)\beta_2 k^2\right]b_2=0,$$
$$\left[-c+\alpha_1 a_0-4\left(1+m^2\right)\beta_1 k^2\right]a_2+2\gamma b_0 b_2=0,$$
$$2\left[-c+\alpha_2 a_0-4\left(\frac{6}{\delta}-1\right)\left(1+m^2\right)\beta_2 k^2\right]b_0 b_2+2\left(\frac{2}{\delta}-1\right)\left(\frac{4}{\delta}-1\right)\beta_2 k^2 b_2^2+\left(\alpha_2 a_2+12m^2\beta_2 k^2\right)b_0^2=0 \tag{22}$$

from which $a_0$, $b_0$, $k$ and $c$ can be determined.

Thus, the periodic solution to the generalized DS equations are

$$u=a_0+a_2\operatorname{sn}^2\xi=(a_0+a_2)-a_2\operatorname{cn}^2\xi=a_0+\frac{a_2}{m^2}-\frac{a_2}{m^2}\operatorname{dn}^2\xi, \tag{23a}$$

$$v=w^{2/\delta}=\left(b_0+b_2\operatorname{sn}^2\xi\right)^{2/\delta}=\left[(b_0+b_2)-b_2\operatorname{cn}^2\xi\right]^{2/\delta}=\left[b_0+\frac{b_2}{m^2}-\frac{b_2}{m^2}\operatorname{dn}^2\xi\right]^{2/\delta}. \tag{23b}$$

When $m\to 1$, Eqs. (23) reduce to

$$u=(a_0+a_2)-a_2\operatorname{sech}^2\xi, \qquad v=\left[(b_0+b_2)-b_2\operatorname{sech}^2\xi\right]^{2/\delta}. \tag{24}$$

## 5. Conclusion

In this Letter, we apply the Jacobi elliptic function expansion to solve three coupled nonlinear systems, there many periodic wave solutions and shock wave or solitary wave solutions are derived. These solutions are helpful in understanding the problems in modern physics.

## Acknowledgement

Many thanks are due to supports from National Natural Science Foundation of China (No. 40305006).

## References

[1] J.L. Hu, Phys. Lett. A 325 (2004) 37.
[2] J.L. Hu, Chin. Phys. 13 (3) (2004) 297.
[3] Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, Phys. Lett. A 290 (2001) 72.
[4] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Phys. Lett. A 289 (2001) 69.
[5] E.J. Parkes, B.R. Duffy, P.C. Abbott, Phys. Lett. A 295 (2002) 280.
[6] N.N. Rao, J. Phys. A 22 (1989) 4813.
[7] X.Y. Wang, B.C. Xu, P.L. Taylor, Phys. Lett. A 173 (1993) 30.
[8] F. Bowman, Introduction to Elliptic Functions with Applications, Universities, London, 1959.
[9] S.K. Liu, S.D. Liu, Nonlinear Equations in Physics, Peking Univ. Press, Beijing, 2000.
[10] V. Prasolov, Y. Solovyev, Elliptic Functions and Elliptic Integrals, American Mathematical Society, Providence, RI, 1997.
[11] Z.X. Wang, D.R. Guo, Special Functions, World Scientific, Singapore, 1989.
[12] L. Boya, J. Casahorran, Phys. Rev. A 39 (1989) 4298.
[13] R. Rajaraman, Phys. Rev. Lett. 42 (1979) 200.
[14] M. Gurses, A. Karasu, Phys. Lett. A 251 (1999) 247.