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Homoclinic (Heteroclinic) Orbit of Complex Dynamical System and Spiral Structure*

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Abstract *Starting from iterated systems, it is shown that the homoclinic (heteroclinic) orbit is a kind of spiral structure. The emphasis is laid to show that there are homoclinic or heteroclinic orbits in complex discrete and continuous systems, and these homoclinic or heteroclinic orbits are some kind of spiral structure.*

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1 Introduction

The spirals are general phenomena in nature.[1,2] But it is difficult to obtain analytical solution for these spiral in dynamical system.[3]. The spiral wave solution of complex Ginzburg–Landau equation has been derived.[4] It is corresponding to heteroclinic orbit between the focus and limit cycle in Ginzburg–Landau equation. Early in 1994, we proposed that the homoclinc or heteroclinic orbit of ordinary differential equation corresponds to solitary wave or wave front of partial differential equations, respectively.[5] In this paper, we show that the relation between spiral and heteroclinic orbit of complex dynamical system.

2 Homoclinic (Heteroclinic) Orbit of Iterated Functional System

In 1989, Prof. Hao pointed out that the homoclinic orbit exists in iterated functional systems.[6] The logistic map

$$x_{n+1} = f(x_n) = 4x_n(1 - x_n) \tag{1}$$

as shown in Fig. 1 has two unstable fixed points $x = 0$ and $x = 3/4$. Hence, staring from $x_0 \neq 0$, for example $x_0 = 1/2$, then $f(x_0)$, $f^2(x_0)$, $f^3(x_0)$, ... approach $x = 0$. And the backward map $f^{-1}(x_0)$, $f^{-2}(x_0)$, $f^{-3}(x_0)$, $\cdots$ approach $x = 0$, too. Therefore, the orbit from iterative sequence forms,

$$\ldots, f^{-3}(x_0), f^{-2}(x_0), f^{-1}(x_0), f(x_0), \\ f^2(x_0), f^3(x_0), \ldots \tag{2}$$

This is a homoclinic orbit which approaches $x = 0$ under the limit $n \to \pm\infty$. In Fig. 1, it is denoted by thick lines with arrows. In fact, the set of homoclinic orbit sequence forms a closed vortex.

If a spiral is denoted by a complex number, the amplitude of complex number varies with rotational angle. There is a complex map (complex iterative functional system),

$$Z_{n+1} = f(Z_n) = \mu Z_n \,, \tag{3}$$

where $Z = x + \mathrm{i}\, y$ is a complex variable, $\mu = a + \mathrm{i}b$ is a complex number.

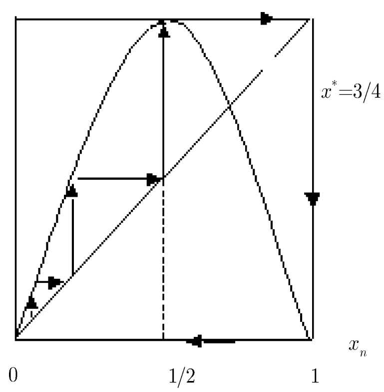

Fig. 1 The homoclinic orbit in map Eq. (1).

Starting from an initial complex number Z_0, the amplitudes of iterative value Z_n increase with iterative number n, when $|\mu| > 1$; the amplitudes of iterative value Z_n decrease with iterative number, when $|\mu| < 1$. Hence, the iterative sequence forms a spiral in complex plane (x, y).

For example, $\mu = (1 + 0.4\mathrm{i})$, the map (3) is

$$Z_{n+1} = (1 + 0.4\mathrm{i})Z_n = f(Z_n) \,, \tag{4}$$

which has two fixed points,[7]

$$Z = 0 \quad \text{and} \quad Z = \infty \,. \tag{5}$$

Hence starting from any point $Z_0 \neq 0$, the amplitudes $(\sqrt{1 + 0.4^2})^n$ of iterative sequence $f(Z_0)$, $f^2(Z_0)$, $f^3(Z_0)$, ..., increase with n. The set of sequence forms a spiral whose amplitude increases with n. And the orbit of sequence approaches fixed point $Z = \infty$. Inversely, the amplitudes $(\sqrt{1 + 0.4^2})^{-n}$ of iterative sequence $f^{-1}(Z_0)$,

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$f^{-2}(Z_0)$, $f^{-3}(Z_0)$, $\ldots$, decrease with n, the sequence approaches another fixed point $Z = 0$. Therefore, the sequence approaches $Z = \infty$ and $Z = 0$ when $n \to \pm\infty$. That is, the heteroclinic orbit forms between $Z = \infty$ and $Z = 0$.

Because fixed point $Z = \infty$ in complex plane (x, y) corresponds to North Pole in Riemann sphere, fixed point $Z = 0$ corresponds to South Pole in Riemann sphere. South Pole is a source of the heteroclinic orbit, North Pole is a sink.[7]

So, the heteroclinic orbit between $Z = \infty$ and $Z = 0$ in complex plane (x, y) corresponds to a sphere spiral as shown in Fig. 2.

Fig. 2 Sphere spiral.[7]

The sphere spirals are called loxodromes in Cartographers. The above analysis shows that the heteroclinic orbit of complex map (4) is connected with spiral structure.

3 Vortex and Spiral of Complex Continuous Dynamical System

A linear complex differential equation is

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \mu Z \tag{6}$$

with $Z = x + \mathrm{i}y$ and $\mu = a + \mathrm{i}b$.

Obviously, $Z = 0$ is a fixed point of equation (6). The solution to equation (6) is

$$Z = Z_0\,\mathrm{e}^{\mu t} = Z_0\,\mathrm{e}^{(a+\mathrm{i}b)t} = Z_0\,\mathrm{e}^{at}\,\mathrm{e}^{\mathrm{i}bt}\,. \tag{7}$$

If $a > 0$, equation (7) shows that $Z(t)$ is a spiral whose amplitude increases with time t. If $a < 0$, $Z(t)$ is a spiral whose amplitude decreases with time t.

A nonlinear complex dynamical system is

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \mu(Z + 1)(Z - 1) \tag{8}$$

with $\mu = a + \mathrm{i}b$.

Equation (8) has two fixed points,

$$Z = 1 \quad \text{and} \quad Z = -1\,. \tag{9}$$

Integrating Eq. (8), we have

$$\frac{1}{2}\ln\frac{2Z - 2}{2Z + 2} = \mu t\,, \tag{10}$$

so, the solution to Eq. (8) is

$$Z = \frac{1 + \mathrm{e}^{2\mu t}}{1 - \mathrm{e}^{2\mu t}} = -\coth(\mu t)\,. \tag{11}$$

From Eq. (11), it is shown that $Z \to -1$, when $t \to +\infty$; $Z \to +1$, when $t \to -\infty$, as $a > 0$. And the behavior approaching the two fixed points is a spiral. So a heteroclinic orbit between the focus point $Z = 1$ and focus point $Z = -1$ in complex plane $Z = X + \mathrm{i}Y$ (where $\mathrm{i} \equiv \sqrt{-1}$, and X and Y are real) is formed, as shown in Fig. 3.

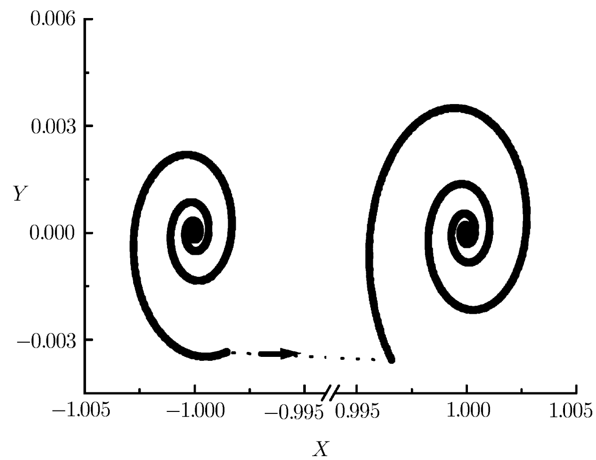

Fig. 3 The focus-focus heteroclinic orbit of complex differential equation (8).

This is a double-spiral structure.

In real dynamical systems, deriving analytical representation of focus-saddle, node-focus and focus-focus heteroclinic orbit is very difficult. But, for complex dynamical systems, it is easy to obtain analytical solution of heteroclinic orbit with focus point.

4 Spiral From Saddle-focus or Node-focus Heteroclinic Orbit

A nonlinear two-dimensional real differential equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (a - \beta)(x - x^2 + y^2) - by(1 - 2x)\,,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = b(x - x^2 + y^2) - (a - \beta)y(1 - 2x)\,, \tag{12}$$

where $a < 0$, $b > 0$, $\beta > 0$ are real numbers.

Let $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t = 0$, the two steady solutions of Eq. (12) are

$$A:\ (x, y) = (0, 0), \qquad B:\ (x, y) = (1, 0)\,. \tag{13}$$

For the two steady solutions A and B, the Jacobian matrices on the right-hand side of Eqs. (12) are

$$\begin{pmatrix} (a-\beta) & -b \\ b & (a-\beta) \end{pmatrix}_A , \quad \begin{pmatrix} -(a-\beta) & -b \\ -b & -(a-\beta) \end{pmatrix}_B , \tag{14}$$

and the corresponding eigenvalues are

$$\lambda = (a-\beta) \pm \mathrm{i}b \quad \text{and} \quad \lambda = -(a-\beta) \pm b\,. \tag{15}$$

Hence the fixed point A is a stable focus, fixed point B is an unstable node ($|a-\beta| > b$) or saddle ($|a-\beta| < b$). The heteroclinic orbit is formed between saddle point B and focus point A, or node point B and focus point A. But it is very difficult to obtain the analytical solution of the heteroclinic orbit from equation (12).

In order to obtain the analytical solution of heteroclinic orbit, we assume $Z = x + \mathrm{i}\,y$. The real dynamical system (12) are reduced to complex dynamical system,

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = (\mu - \beta)Z(1-Z)\,, \tag{16}$$

where $\mu = a + \mathrm{i}b$, $a < 0$, $\beta > 0$.

The complex dynamical system (16) is much simpler than real dynamical system (12).

From Eq. (16), we have

$$\frac{\mathrm{d}Z}{Z(1-Z)} = (\mu - \beta)\mathrm{d}t\,. \tag{17}$$

Integrating Eq. (17), we obtain

$$1 - 2Z = \tanh\Big(\frac{\beta-\mu}{2}\Big)t\,. \tag{18}$$

From Eq. (18), it is obvious that $Z \to 0$ with spiral behavior when $t \to +\infty$; $Z \to 1$, when $t \to -\infty$.

Therefore equation (18) is a heteroclinic orbit solution of saddle-focus or node-focus. Separating Eq. (18) into real part and imaginary part, we have

$$x = \frac{1 + \mathrm{e}^{-(a-\beta)t}\cos(bt)}{1 + 2\,\mathrm{e}^{-(a-\beta)t}\cos(bt) + \mathrm{e}^{-2(a-\beta)t}}\,,$$

$$y = \frac{\mathrm{e}^{-(a-\beta)t}\sin(bt)}{1 + 2\,\mathrm{e}^{-(a-\beta)t}\cos(bt) + \mathrm{e}^{-2(a-\beta)t}}\,. \tag{19}$$

Equations (19) are also the saddle-focus (or node-focus) heteroclinic orbit solution. They are all spiral structure.

5 Conclusion

Starting from iterated system, it is shown that homoclinic (heteroclinic) orbit is a kind of spiral structure. In this paper, the emphasis is laid to show that there are homoclinic or heteroclinic orbits in complex discrete and continuous systems, and these homoclinic or heteroclinic orbits are some kind of spiral structure. The difference between these spiral structures is due to the existence of different fixed points and different controlling systems. Of course, there are still many things deserving further research.

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