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Exact Solutions to Degasperis–Procesi Equation*

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Abstract *In this paper, dependent and independent variable transformations are introduced to solve the Degasperis–Procesi equation. It is shown that different kinds of solutions can be obtained to the Degasperis–Procesi equation.*

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1 Introduction

The family of equations

$$u_t - u_{txx} + (b+2)uu_x = (b+1)u_x u_{xx} + uu_{xxx}\,, \tag{1}$$

where $b > 0$ is a constant, was discussed in Ref. [1], and phase portraits were applied to categorize traveling wave solutions. It is known that there exist only two kinds of integrable equations for this family of equations, namely, the dispersionless Camassa–Holm equation (CHE for short),[2]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}\,, \tag{2}$$

for $b = 1$ and the Degasperis–Procesi equation (DPE for short),[3]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}\,, \tag{3}$$

for $b = 2$.

Usually, equation (1) is difficult to be solved, some transformations have to be introduced. For example, Vakhnenko *et al.*[4] introduced a new dependent variable z,

$$z = \frac{u-v}{|v|}\,, \tag{4}$$

and assumed that z is an implicit or explicit function of η, where

$$\eta = x - vt - x_0\,, \tag{5}$$

v and x_0 are arbitrary constants and $v \neq 0$.

Through the above transformations, they obtained hump-like, inverted loop-like, and coshoidal periodic wave solutions, hump-like, inverted loop-like and peakon solitary wave solutions to Eq. (3).

By introducing transformations, Matsuno[5,6] found the N-soliton solutions and multi-soliton solutions to DPE.

Since DPE is a current research interest in soliton theory, in this paper, based on the introduced transformations, we will show systematical results for the DPE (3) by using the knowledge of elliptic equation and Jacobian elliptic functions.[7−11]

2 Transformed Equation (1)

In order to solve the DPE, certain dependent or independent variable transformations must be introduced. Starting from Eq. (1), we define

$$x = w(y,\tau)\,,\quad t = \tau\,, \tag{6}$$

then we have

$$\frac{\partial}{\partial x} = \frac{1}{w_y}\frac{\partial}{\partial y}\,,\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{w_\tau}{w_y}\frac{\partial}{\partial y}\,. \tag{7}$$

Substituting this transformation into Eq. (1) yields

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} - \frac{w_\tau}{w_y}\frac{\partial u}{\partial y} + (b+2)\frac{u}{w_y}\frac{\partial u}{\partial y} - \frac{1}{w_y^2}\frac{\partial^3 u}{\partial\tau\partial y^2} + \frac{w_\tau}{w_y^3}\frac{\partial^3 u}{\partial y^3} + \frac{2w_{\tau y}w_y - 3w_\tau w_{yy}}{w_y^4}\frac{\partial^2 u}{\partial y^2}\\
&\quad + \frac{w_{\tau yy}w_y^2 - w_\tau w_y w_{yyy} - 3w_{\tau y}w_y w_{yy} + 3w_\tau w_{yy}^2}{w_y^5}\frac{\partial u}{\partial y}\\
&= (b+1)\frac{1}{w_y}\frac{\partial u}{\partial y}\Big(\frac{1}{w_y^2}\frac{\partial^2 u}{\partial y^2} - \frac{w_{yy}}{w_y^3}\frac{\partial u}{\partial y}\Big) + u\Big(\frac{1}{w_y^3}\frac{\partial^3 u}{\partial y^3} - \frac{3w_{yy}}{w_y^4}\frac{\partial^2 u}{\partial y^2} - \frac{w_y w_{yyy} - 3w_{yy}^2}{w_y^5}\frac{\partial u}{\partial y}\Big)\,.
\end{aligned} \tag{8}$$

from which one has

$$\begin{aligned}
&w_y^5\frac{\partial u}{\partial \tau} - w_\tau w_y^4\frac{\partial u}{\partial y} + (b+2)w_y^4 u\frac{\partial u}{\partial y} - w_y^3\frac{\partial^3 u}{\partial\tau\partial y^2} + w_\tau w_y^2\frac{\partial^3 u}{\partial y^3} + 2w_y^2 w_{\tau y}\frac{\partial^2 u}{\partial y^2} - 3w_\tau w_y w_{yy}\frac{\partial^2 u}{\partial y^2}\\
&\quad + (w_{\tau yy}w_y^2 - w_\tau w_y w_{yyy} - 3w_{\tau y}w_y w_{yy} + 3w_\tau w_{yy}^2)\frac{\partial u}{\partial y}
\end{aligned}$$

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$$= (b+1)w_y^2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} - (b+1)w_y w_{yy}\Big(\frac{\partial u}{\partial y}\Big)^2 + w_y^2 u\frac{\partial^3 u}{\partial y^3} - 3w_y w_{yy} u\frac{\partial^2 u}{\partial y^2} - w_y w_{yyy} u\frac{\partial u}{\partial y} + 3w_{yy}^2 u\frac{\partial u}{\partial y}\,. \tag{9}$$

If we set

$$u(x,t) = R(y,\tau)\,, \tag{10}$$

then from Eq. (9) we have

$$\begin{aligned} &w_y^5 R_\tau - w_\tau w_y^4 R_y + (b+2)w_y^4 R R_y - w_y^3 R_{\tau yy} \\ &\quad + w_\tau w_y^2 R_{yyy} + 2w_y^2 w_{\tau y} R_{yy} - 3w_\tau w_y w_{yy} R_{yy} \\ &\quad + (w_{\tau yy} w_y^2 - w_\tau w_y w_{yyy} - 3w_{\tau y} w_y w_{yy} + 3w_\tau w_{yy}^2) R_y \\ &= (b+1)w_y^2 R_y R_{yy} - (b+1) w_y w_{yy} R_y^2 + w_y^2 R R_{yyy} \\ &\quad - 3w_y w_{yy} R R_{yy} - w_y w_{yyy} R R_y + 3w_{yy}^2 R R_y\,. \end{aligned} \tag{11}$$

If we choose

$$R(y,\tau) = w_\tau(y,\tau)\,, \tag{12}$$

for $b=2$, i.e. for DPE, we have

$$w_y^2 w_{\tau\tau} + 3w_y w_\tau w_{\tau y} - w_{\tau\tau yy} = 0\,, \tag{13}$$

this transformed equation can be easily solved to derive its traveling wave solutions.

3 Exact Traveling Wave Solutions to DPE

In this section, we will try ro find the traveling wave solutions to transformed DPE (13). In doing so, we first take the transformation in the following frame

$$\xi = k(y - c\tau)\,, \tag{14}$$

where k is wave number and c is wave speed.

Combining Eq. (14) with Eq. (13) leads to

$$w_{\xi\xi\xi\xi} - 4w_\xi^2 w_{\xi\xi} = 0\,. \tag{15}$$

In fact, equation (15) is still not easy to be solved by the usual function expansion methods, such as Jacobi elliptic function expansion methods,[7,8] for the expansion rank of Eq. (15) is zero.

In order to solve Eq. (15) exactly, we set

$$V = w_\xi\,, \tag{16}$$

then equation (15) can be rewritten as

$$V_{\xi\xi\xi} - 4VV_\xi = 0\,, \tag{17}$$

which can be integrated twice and rewritten as

$$V_\xi^2 = \frac{2}{3}V^4 + AV + B \equiv P(V)\,, \tag{18}$$

where A and B are two integration constants.

If $P(V)=0$ has four real roots, i.e.,

$$(V-\alpha)(V-\beta)(V-\gamma)(V-\delta) = \frac{2}{3}V^4 + AV + B = 0\,, \tag{19}$$

with

$$\begin{aligned} &\alpha \ge \beta \ge \gamma \ge \delta\,, \quad \alpha+\beta+\gamma+\delta = 0\,, \\ &\alpha\beta + (\alpha+\beta)(\gamma+\delta) = 0\,, \\ &A = -\frac{2}{3}[\alpha\beta(\gamma+\delta) + (\alpha+\beta)\gamma\delta]\,, \end{aligned}$$

$$B = \frac{2}{3}\alpha\beta\gamma\delta\,. \tag{20}$$

If $V > \alpha$ or $V < \delta$, we have $P(V) > 0$, so we can set[12]

$$\frac{V-\alpha}{V-\beta} = \frac{\alpha-\delta}{\beta-\delta}Z^2\,, \quad m^2 \equiv \frac{(\alpha-\delta)(\beta-\gamma)}{(\alpha-\gamma)(\beta-\delta)}\,, \tag{21}$$

where $0 \le m \le 1$ is called modulus of Jacobi elliptic functions.[7–11]

From Eqs. (18) and (21), we have

$$\mathrm{d}V = \frac{2}{\alpha-\beta}\sqrt{\frac{\alpha-\delta}{\beta-\delta}}\,(V-\beta)\sqrt{(V-\alpha)(V-\beta)}\,\mathrm{d}Z\,, \tag{22}$$

and

$$\mathrm{d}\xi = \frac{\mathrm{d}V}{\sqrt{P(V)}} = \frac{2\sqrt{3}}{\sqrt{2(\alpha-\gamma)(\beta-\delta)}}\frac{\mathrm{d}Z}{\sqrt{\ell(Z)}} \tag{23}$$

with

$$\ell(Z) \equiv (1-Z^2)(1-m^2Z^2)\,. \tag{24}$$

Thus we have

$$Z = \mathrm{sn}(k\xi, m)\,, \quad k = \sqrt{\frac{(\alpha-\gamma)(\beta-\delta)}{6}}\,, \tag{25}$$

and

$$V = \frac{\beta(\alpha-\delta)\,\mathrm{sn}^2(k\xi,m) - \alpha(\beta-\delta)}{(\alpha-\delta)\,\mathrm{sn}^2(k\xi,m) - (\beta-\delta)}\,, \tag{26}$$

where $\mathrm{sn}(k\xi,m)$ is the Jacobi sine elliptic function.[9–11]

When $\gamma \le V \le \beta$, we have $P(V) \ge 0$, so we define

$$\frac{V-\gamma}{V-\delta} = \frac{\beta-\gamma}{\beta-\delta}Z^2\,, \quad m^2 \equiv \frac{(\alpha-\delta)(\beta-\gamma)}{(\alpha-\gamma)(\beta-\delta)}\,, \tag{27}$$

from which we have

$$\frac{\mathrm{d}V}{\sqrt{P(V)}} = \frac{2\sqrt{3}}{\sqrt{2(\alpha-\gamma)(\beta-\delta)}}\frac{\mathrm{d}Z}{\sqrt{\ell(Z)}}\,, \tag{28}$$

so we can have the same result as Eqs. (25) and (26).

From Eqs. (25) and (26), the solution to DPE can be determined as

$$\begin{aligned} u &= -kc\,\frac{\beta(\alpha-\delta)\,\mathrm{sn}^2(k\xi,m) - \alpha(\beta-\delta)}{(\alpha-\delta)\,\mathrm{sn}^2(k\xi,m) - (\beta-\delta)}\,, \\ k &= \sqrt{\frac{(\alpha-\gamma)(\beta-\delta)}{6}}\,, \end{aligned} \tag{29}$$

and

$$\begin{aligned} x = x_0 &+ \frac{\alpha}{k}\Pi\Big(\varphi, \frac{\alpha-\delta}{\beta-\delta}, m\Big) - \frac{\beta-\delta}{k(\alpha-\delta)} \\ &\times \Big[\Pi\Big(\varphi, \frac{\alpha-\delta}{\beta-\delta}, m\Big) - F(\varphi, m)\Big] \end{aligned} \tag{30}$$

with

$$\varphi = \sin^{-1}\,\mathrm{sn}(k\xi)\,, \tag{31}$$

and

$$F(\varphi, m) = \int_0^\varphi \frac{\mathrm{d}\theta}{\sqrt{1-m^2\sin^2\theta}}\,, \tag{32}$$

called the normal elliptic integral of the first kind, and

$$\Pi(\varphi,\lambda^2,m)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{(1-\lambda^2\sin^2\theta)\sqrt{1-m^2\sin^2\theta}}\,,\quad -\infty<\lambda^2<\infty\,,\tag{33}$$

called the normal elliptic integral of the third kind.[13]

Obviously, the solution of DPEs (29) and (30) has not been reported in the literature.

For Eq. (18), there is a special case needed to be considered, i.e. when $A=B=0$, then we have

$$V_\xi=\pm\sqrt{\frac{2}{3}}V^2\,,\tag{34}$$

from which one has

$$V=\frac{1}{V_0\pm\sqrt{2/3}(\xi-\xi_0)}\,,\tag{35}$$

where V_0 and ξ_0 are two constants.

From Eq. (35), the solution to DPE can be defined as

$$u=-\frac{kc}{V_0\pm\sqrt{2/3}(\xi-\xi_0)}\,,\tag{36}$$

and

$$x=x_0\pm\sqrt{\frac{3}{2}}kc\ln\Big|\pm\sqrt{\frac{2}{3}}(\xi-\xi_0)+V_0\Big|\,,\tag{37}$$

where x_0 is a constant.

4 Conclusion

In this paper, we presented the process to find exact solutions for the DPE and obtained some new types of solutions. These solutions may be applied to describe and/or explain some phenomena found in the nature, such as water waves, since the model such as DPE or CHE has been proposed to model these phenomena. Due to the solutions presented in this paper are just some special solutions, more methods are still needed to find more types of solutions to DPE.

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