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Two kinds of multi-order exact solution to the shallow water system

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Abstract

In this paper, two kinds of Lamé function have been listed and compared. Based on these two Lamé functions and the Jacobi elliptic function, the perturbation method is applied to the shallow water system, and many multi-order solutions of novel form are derived. In addition, it is shown that different Lamé functions can exist in the first-order solutions of the nonlinear system.

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1. Two kinds of Lamé function

Usually, the Lamé equation [1, 2] in terms of $y(x)$ can be written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}\eta^2}+\frac{1}{2}\left(\frac{1}{\eta}+\frac{1}{\eta-1}+\frac{1}{\eta-h}\right)\frac{\mathrm{d}y}{\mathrm{d}\eta}-\frac{\mu+n(n+1)\eta}{4\eta(\eta-1)(\eta-h)}y=0, \quad (1)$$

where

$$h=m^{-2}>1,\qquad \mu=-h\lambda, \quad (2)$$

where λ is an eigenvalue, n is a positive integer and m $(0<m<1)$ is the modulus of the Jacobi elliptic function.

Equation (1) is a kind of Fuchs-type equation with four regular singular points $\eta=0,1,h$ and $\eta=\infty$; the solution to Lamé equation (1) is known as the Lamé function. When η takes different forms, there will be a Lamé equation with different forms whose solutions will be different Lamé functions. For example, if we set

$$\eta=\mathrm{sn}^2x, \quad (3)$$

then Lamé equation (1) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left[\lambda-n(n+1)m^2\mathrm{sn}^2x\right]y=0, \quad (4)$$

where $\mathrm{sn}\,x$ is the Jacobi elliptic sine function with modulus m $(0<m<1)$.

For equation (4), there are different Lamé functions expressed in closed form; for example, when $n=3$, $\lambda=4(1+m^2)$, i.e. $\mu=-4(1+m^{-2})$, the Lamé function is

$$L_3^{\mathrm{sn}}(x)=\eta^{1/2}(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x. \quad (5)$$

For $n=2$, when $\lambda=1+m^2$, the Lamé function is

$$L_2^{\mathrm{s}}(x)=(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{cn}\,x\,\mathrm{dn}\,x, \quad (6)$$

when $\lambda=1+4m^2$, the Lamé function is

$$L_2^{\mathrm{c}}(x)=(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{sn}\,x\,\mathrm{dn}\,x \quad (7)$$

and when $\lambda=4+m^2$, the Lamé function is

$$L_2^{\mathrm{d}}(x)=(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{sn}\,x\,\mathrm{cn}\,x. \quad (8)$$

In equations (5)–(8), $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind [1, 2], respectively.

Could equation (1) be written in other forms and will they have different solutions similar to those given above for equation (4)? In fact, if we set

$$\eta=\mathrm{cd}^2x, \quad (9)$$

then Lamé equation (1) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left[\lambda-n(n+1)m^2\,\mathrm{cd}^2x\right]y=0, \quad (10)$$

where $\mathrm{cd}\,x\equiv\mathrm{cn}\,x/\mathrm{dn}\,x$ is another kind of Jacobi elliptic function with modulus m $(0<m<1)$.

For equation (10), there also exist different Lamé functions expressed in closed form; for example, when $n = 3$, $\lambda = 4(1+m^2)$, i.e. $\mu = -4(1+m^{-2})$, the Lamé function is

$$L_3^{\mathrm{cd}}(x) = \eta^{1/2}(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x. \quad (11)$$

This is another Lamé function different from that given in (5).

For $n = 2$, when $\lambda = 1+m^2$, the Lamé function is

$$L_2^{\mathrm{cd}}(x) = (1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{sd}\,x\,\mathrm{nd}\,x, \quad (12)$$

when $\lambda = 1+4m^2$, the Lamé function is

$$L_2^{\mathrm{sd}}(x) = (1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{cd}\,x\,\mathrm{nd}\,x \quad (13)$$

and when $\lambda = 4+m^2$, the Lamé function is

$$L_2^{\mathrm{nd}}(x) = (1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2} = \mathrm{sd}\,x\,\mathrm{cd}\,x. \quad (14)$$

In equations (11)–(14), $\mathrm{sd}\,x \equiv \mathrm{sn}\,x/\mathrm{dn}\,x$, $\mathrm{nd}\,x \equiv 1/\mathrm{dn}\,x$ are two new Jacobi elliptic functions. $L_2^{\mathrm{cd}}(x)$, $L_2^{\mathrm{sd}}(x)$ and $L_2^{\mathrm{nd}}(x)$ are three new Lamé functions different from those given in (6), (7) and (8).

During the past three decades, nonlinear wave research has made great progress, and a number of new methods have been proposed to obtain exact solutions to nonlinear wave equations. Among these methods, the homogeneous balance method [3], the hyperbolic tangent function expansion method [4, 5], the nonlinear transformation method [6, 7], the trial function method [8, 9], the sine–cosine method [10], the Jacobi elliptic function expansion method [11–13], the auxiliary equation and mapping method [14], the exp-function method [15] and so on have been widely applied to solve nonlinear wave equations exactly. Furthermore, in order to discuss the stability of these solutions, one must superimpose a small disturbance on them and analyze the evolution of the small disturbance [2, 16]. This is equivalent to solutions of nonlinear evolution equations expanded as a power series in terms of a small parameter ϵ, and then multi-order exact solutions are derived. Lamé functions given by (5), (6), (7) and (8), i.e. $L_3^{\mathrm{sn}}(x)$ and/or $L_2^{\mathrm{s}}(x)$ and/or $L_2^{\mathrm{c}}(x)$ and/or $L_2^{\mathrm{d}}(x)$, have been applied to solve many single nonlinear equations to derive multi-order solutions [17]. Could Lamé functions given by (11), (12), (13) and (14), i.e. $L_3^{\mathrm{cd}}(x)$, $L_2^{\mathrm{cd}}(x)$, $L_2^{\mathrm{sd}}(x)$ and $L_2^{\mathrm{nd}}(x)$, be applied to solve nonlinear equations? We will answer this question in the following sections. Since we have listed two kinds of different Lamé functions, $L_3^{\mathrm{sn}}(x)$, $L_2^{\mathrm{s}}(x)$, $L_2^{\mathrm{c}}(x)$, $L_2^{\mathrm{d}}(x)$ and $L_3^{\mathrm{cd}}(x)$, $L_2^{\mathrm{cd}}(x)$, $L_2^{\mathrm{sd}}(x)$, $L_2^{\mathrm{nd}}(x)$, we will apply these two kinds of Lamé function to the shallow water system to illustrate the applications of Lamé functions to nonlinear equations in order to derive different kinds of multi-order exact solutions.

2. Shallow water system and its perturbed expansion

The generalized shallow water system reads

$$H_t + (Hu)_x + \beta u_{xxx} = 0, \quad (15a)$$

$$u_t + H_x + uu_x = 0. \quad (15b)$$

We seek travelling wave solutions of the following form:

$$u = u(\xi), \quad H = H(\xi), \quad \xi = k(x-ct), \quad (16)$$

where k and c are wave number and wave speed, respectively.

Substituting (16) into (15) yields

$$-cH_\xi + (Hu)_\xi + \beta k^2 u_{\xi\xi\xi} = 0, \quad (17a)$$

$$-cu_\xi + H_\xi + uu_\xi = 0, \quad (17b)$$

which can be integrated as

$$-cH + Hu + \beta k^2 u_{\xi\xi} = C_1, \quad (18a)$$

$$-cu + H + \tfrac{1}{2}u^2 = C_2, \quad (18b)$$

where C_1 and C_2 are two integration constants. We will see that not all integration constants can be taken to be zero.

Applying the perturbation method and setting

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \quad (19a)$$

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots, \quad (19b)$$

where ϵ $(0 < \epsilon << 1)$ is a small parameter, u_0, u_1, u_2 and H_0, H_1, H_2 represent the zeroth-, first- and second-order solutions, respectively.

Substituting (19) into (18), we obtain various order equations. For example, the zeroth-order equation (for ϵ^0) is

$$-cH_0 + H_0u_0 + \beta k^2 u_{0\xi\xi} = C_1, \quad (20a)$$

$$-cu_0 + H_0 + \tfrac{1}{2}u_0^2 = C_2, \quad (20b)$$

the first-order equation (for ϵ^1) is

$$-cH_1 + H_0u_1 + H_1u_0 + \beta k^2 u_{1\xi\xi} = 0, \quad (21a)$$

$$-cu_1 + H_1 + u_0u_1 = 0, \quad (21b)$$

and the second-order equation (ϵ^2) is

$$-cH_2 + H_0u_2 + H_1u_1 + H_2u_0 + \beta k^2 u_{2\xi\xi} = 0, \quad (22a)$$

$$-cu_2 + H_2 + \tfrac{1}{2}u_1^2 + u_0u_2 = 0. \quad (22b)$$

For the zeroth-order equation (20), the Jacobi elliptic function expansion method [11, 12] can be applied to solve it. In fact, different Jacobi elliptic functions can be applied to solve (20), and different results can be derived. In the following, we show that different zeroth-order solutions will allow us to have different first-order and second-order solutions, where these solutions can be represented in terms of different Lamé functions shown in the above sections.

3. $L_3^{\mathrm{sn}}(x)$, $L_2^{\mathrm{s}}(x)$, $L_2^{\mathrm{c}}(x)$, $L_2^{\mathrm{d}}(x)$ and the first kind of multi-order solution

3.1. Jacobi elliptic sine function expansion and Lamé function solution

For the zeroth-order equation (20), the Jacobi elliptic sine function expansion method [11, 12] can be applied to solve it, i.e. the ansatz solution is supposed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{sn}\,\xi + a_2\,\mathrm{sn}^2\,\xi, \quad (23a)$$

$$u_0 = b_0 + b_1\,\mathrm{sn}\,\xi, \quad (23b)$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (23) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0, \quad C_2 = -\frac{c^2}{2} + \beta k^2(1+m^2), \tag{24a}$$

$$b_0 = c, \quad b_1 = \pm 2mk\sqrt{\beta}. \tag{24b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(1+m^2) - 2m^2k^2\beta\,\mathrm{sn}^2\,\xi, \tag{25a}$$

$$u_0 = c \pm 2mk\sqrt{\beta}\,\mathrm{sn}\,\xi. \tag{25b}$$

Substituting (25) and (24*a*) into the first-order equation (21) yields

$$H_1 = (c-u_0)u_1, \tag{26a}$$

$$\beta k^2 \frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + H_0 u_1 + (u_0 - c)H_1 = 0, \tag{26b}$$

from which we have

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + \left[(1+m^2) - 6m^2\,\mathrm{sn}^2\,\xi\right]u_1 = 0. \tag{27}$$

Here it is obvious that u_1 in (27) takes the same form as y in (4). Hence we can assume that u_1 takes the following form:

$$u_1 = A_\mathrm{s}L_2^\mathrm{s}(\xi) = A_\mathrm{s}\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi. \tag{28}$$

Substituting (28) into (26*a*) yields

$$H_1 = \mp 2mk\sqrt{\beta}A_\mathrm{s}L_3^\mathrm{sn}(\xi). \tag{29}$$

Hence the final first-order solution is

$$H_1 = \mp 2mk\sqrt{\beta}A_\mathrm{s}L_3^\mathrm{sn}(\xi), \quad u_1 = A_\mathrm{s}L_2^\mathrm{s}(\xi), \tag{30}$$

where A_s is an arbitrary constant; obviously different Lamé functions $L_3^\mathrm{sn}(\xi)$ and $L_2^\mathrm{s}(\xi)$ exist in the same system for the shallow water system.

In order to obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (25) and the first-order solution (30) back into the second-order equation (22). We have

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{31a}$$

$$\beta k^2 \frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + H_0 u_2 + (u_0 - c)H_2 + H_1 u_1 = 0, \tag{31b}$$

from which we have

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + \left[(1+m^2) - 6m^2\,\mathrm{sn}^2\,\xi\right]u_2 = \pm\frac{3m}{k\sqrt{\beta}}A_\mathrm{s}^2\mathrm{sn}\,\xi\,\mathrm{cn}^2\,\xi\,\mathrm{dn}^2\,\xi. \tag{32}$$

Since $\mathrm{cn}^2\,\xi = 1 - \mathrm{sn}^2\,\xi$, $\mathrm{dn}^2\,\xi = 1 - m^2\,\mathrm{sn}^2\,\xi$, the special solution to (32) can be assumed to be

$$u_2 = B_1\,\mathrm{sn}\,\xi + B_3\,\mathrm{sn}^3\,\xi. \tag{33}$$

Substituting (33) into (32) yields

$$B_1 = \mp\frac{(1+m^2)A_\mathrm{s}^2}{4mk\sqrt{\beta}}, \quad B_3 = \pm\frac{mA_\mathrm{s}^2}{2k\sqrt{\beta}}, \tag{34}$$

i.e. the second-order solution is

$$u_2 = \mp\frac{(1+m^2)A_\mathrm{s}^2}{4mk\sqrt{\beta}}\mathrm{sn}\,\xi \pm \frac{mA_\mathrm{s}^2}{2k\sqrt{\beta}}\mathrm{sn}^3\,\xi, \tag{35a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{35b}$$

where u_0 and u_1 are given by (25*b*) and (28).

3.2. Jacobi elliptic cosine function expansion and Lamé function solution

For the zeroth-order equation (20), the Jacobi elliptic cosine function expansion method [11, 12] can also be applied to solve it; i.e. the ansatz solution is supposed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{cn}\,\xi + a_2\,\mathrm{cn}^2\,\xi, \tag{36a}$$

$$u_0 = b_0 + b_1\,\mathrm{cn}\,\xi, \tag{36b}$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (36) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0,$$
$$C_2 = -\frac{c^2}{2} + \beta k^2(1-2m^2), \tag{37a}$$

$$b_0 = c, \quad b_1 = \pm 2mk\sqrt{-\beta}. \tag{37b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(1-2m^2) + 2m^2k^2\beta\,\mathrm{cn}^2\,\xi, \tag{38a}$$

$$u_0 = c \pm 2mk\sqrt{-\beta}\,\mathrm{cn}\,\xi. \tag{38b}$$

Substituting (38) and (37*a*) into the first-order equation (21) yields

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + \left[(1-2m^2) + 6m^2\,\mathrm{cn}^2\,\xi\right]u_1 = 0. \tag{39}$$

Here it is obvious that u_1 in (39) takes the same form as y in (4). Hence we can assume that u_1 takes the following form:

$$u_1 = A_\mathrm{c}L_2^\mathrm{c}(\xi) = A_\mathrm{c}\,\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi. \tag{40}$$

Substituting (40) into (26*a*) yields

$$H_1 = \mp 2mk\sqrt{-\beta}A_\mathrm{c}L_3^\mathrm{sn}(\xi). \tag{41}$$

Hence the final first-order solution is

$$H_1 = \mp 2mk\sqrt{\beta}A_\mathrm{s}L_3^\mathrm{sn}(\xi), \quad u_1 = A_\mathrm{c}L_2^\mathrm{c}(\xi), \tag{42}$$

where A_c is an arbitrary constant; obviously different Lamé functions $L_3^\mathrm{sn}(\xi)$ and $L_2^\mathrm{c}(\xi)$ exist in the same system for the shallow water system.

To obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (38) and the first-order solution (42) back into the second-order equation (22). Hence we have

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + \left[(1-2m^2) + 6m^2\,\mathrm{cn}^2\,\xi\right]u_2$$
$$= \mp\frac{3m}{k\sqrt{-\beta}}A_\mathrm{c}^2\,\mathrm{cn}\,\xi\,\mathrm{sn}^2\,\xi\,\mathrm{dn}^2\,\xi. \tag{43}$$

Since $\mathrm{cn}^2\,\xi = 1 - \mathrm{sn}^2\,\xi$, $\mathrm{dn}^2\,\xi = 1 - m^2\mathrm{sn}^2\,\xi$, the special solution to (43) can be assumed to be

$$u_2 = B_1\,\mathrm{cn}\,\xi + B_3\,\mathrm{cn}^3\,\xi. \tag{44}$$

Substituting (44) into (43) yields

$$B_1 = \pm\frac{(2m^2-1)A_c^2}{4mk\sqrt{-\beta}}, \quad B_3 = \mp\frac{mA_c^2}{2k\sqrt{-\beta}}, \tag{45}$$

i.e. the second-order solution is

$$u_2 = \pm\frac{(2m^2-1)A_c^2}{4mk\sqrt{-\beta}}\mathrm{cn}\,\xi \mp \frac{mA_c^2}{2k\sqrt{-\beta}}\mathrm{cn}^3\,\xi, \tag{46a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{46b}$$

where u_0 and u_1 are given by (38*b*) and (40).

3.3. Jacobi elliptic function of the third kind expansion method and Lamé function solution

For the zeroth-order equation (20), the Jacobi elliptic function of the third kind expansion method [11, 12] can also be applied to solve it; i.e. the ansatz solution is supposed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{dn}\,\xi + a_2\,\mathrm{dn}^2\,\xi, \tag{47a}$$

$$u_0 = b_0 + b_1\,\mathrm{dn}\,\xi, \tag{47b}$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (47) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0, \quad C_2 = -\frac{c^2}{2} + \beta k^2(m^2-2), \tag{48a}$$

$$b_0 = c, \quad b_1 = \pm 2k\sqrt{-\beta}. \tag{48b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(m^2-2) + 2k^2\beta\,\mathrm{dn}^2\,\xi, \tag{49a}$$

$$u_0 = c \pm 2k\sqrt{-\beta}\,\mathrm{dn}\,\xi. \tag{49b}$$

Substituting (49) and (48*a*) into the first-order equation (21) yields

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2} + \left[(m^2-2) + 6\,\mathrm{dn}^2\,\xi\right]u_1 = 0. \tag{50}$$

Here it is obvious that u_1 in (50) takes the same form as y in (4). Hence we can assume that u_1 takes the following form:

$$u_1 = A_\mathrm{d}L_2^\mathrm{d}(\xi) = A_\mathrm{d}\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi. \tag{51}$$

Substituting (51) into (26*a*) yields

$$H_1 = \mp 2k\sqrt{-\beta}A_\mathrm{d}L_3^\mathrm{sn}(\xi). \tag{52}$$

Hence the final first-order solution is

$$H_1 = \mp 2k\sqrt{\beta}A_\mathrm{d}L_3^\mathrm{sn}(\xi), \quad u_1 = A_\mathrm{d}sL_2^\mathrm{d}(\xi), \tag{53}$$

where A_d is an arbitrary constant; obviously different Lamé functions $L_3^\mathrm{sn}(\xi)$ and $L_2^\mathrm{d}(\xi)$ exist in the same system for the shallow water system.

In order to obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (49) and the first-order solution (53) back into the second-order equation (22). We have

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + \left[(m^2-2) + 6\,\mathrm{dn}^2\,\xi\right]u_2 = \mp\frac{3}{k\sqrt{-\beta}}A_\mathrm{d}^2\,\mathrm{dn}\,\xi\,\mathrm{sn}^2\,\xi\,\mathrm{cn}^2\,\xi. \tag{54}$$

Since $\mathrm{cn}^2\,\xi = 1-\mathrm{sn}^2\,\xi$, $\mathrm{dn}^2\,\xi = 1-m^2\mathrm{sn}^2\,\xi$, the special solution to (54) can be assumed to be

$$u_2 = B_1\,\mathrm{dn}\,\xi + B_3\,\mathrm{dn}^3\,\xi. \tag{55}$$

Substituting (55) into (54) yields

$$B_1 = \pm\frac{(m^2-2)A_\mathrm{d}^2}{2k\sqrt{-\beta}}, \quad B_3 = \mp\frac{A_\mathrm{d}^2}{2k\sqrt{-\beta}}, \tag{56}$$

i.e. the second-order solution is

$$u_2 = \pm\frac{(m^2-2)A_\mathrm{d}^2}{2k\sqrt{-\beta}}\,\mathrm{dn}\,\xi \mp \frac{A_\mathrm{d}^2}{2k\sqrt{-\beta}}\mathrm{dn}^3\,\xi, \tag{57a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{57b}$$

where u_0 and u_1 are given by (49*b*) and (51).

4. $L_3^\mathrm{cd}(x)$, $L_2^\mathrm{cd}(x)$, $L_2^\mathrm{sd}(x)$, $L_2^\mathrm{nd}(x)$ and the second kind of multi-order solution

4.1. cd ξ expansion and Lamé function solution

For the zeroth-order equation (20), the Jacobi elliptic function expansion method [11, 12] can be applied to solve it; for example, the ansatz solution can be supposed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{cd}\,\xi + a_2\,\mathrm{cd}^2\,\xi, \tag{58a}$$

$$u_0 = b_0 + b_1\,\mathrm{cd}\,\xi, \tag{58b}$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (59) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0, \quad C_2 = -\frac{c^2}{2} + \beta k^2(1+m^2), \tag{59a}$$

$$b_0 = c, \quad b_1 = \pm 2mk\sqrt{\beta}. \tag{59b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(1+m^2) - 2m^2k^2\beta\,\mathrm{cd}^2\,\xi, \tag{60a}$$

$$u_0 = c \pm 2mk\sqrt{\beta}\,\mathrm{cd}\,\xi. \tag{60b}$$

Substituting (60) and (59*a*) into the first-order equation (21) yields

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2} + \left[(1+m^2) - 6m^2\,\mathrm{cd}^2\,\xi\right]u_1 = 0. \tag{61}$$

Here it is obvious that u_1 in (61) takes the same form as y in (10). Hence we can assume that u_1 takes the following form:

$$u_1 = A_\mathrm{cd}L_2^\mathrm{cd}(\xi) = A_\mathrm{cd}\,\mathrm{sd}\,\xi\,\mathrm{nd}\,\xi. \tag{62}$$

Substituting (62) into (26*a*) yields

$$H_1 = \mp 2mk\sqrt{\beta}A_{\mathrm{cd}}L_3^{\mathrm{cd}}(\xi). \tag{63}$$

Hence the final first-order solution is

$$H_1 = \mp 2mk\sqrt{\beta}A_{\mathrm{cd}}L_3^{\mathrm{cd}}(\xi), \quad u_1 = A_{\mathrm{cd}}L_2^{\mathrm{cd}}(\xi), \tag{64}$$

where A_{cd} is an arbitrary constant; obviously different Lamé functions $L_3^{\mathrm{cd}}(\xi)$ and $L_2^{\mathrm{cd}}(\xi)$ exist in the same system for the shallow water system.

In order to obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (60) and the first-order solution (64) back into the second-order equation (22). We have

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{65a}$$

$$\beta k^2\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + H_0u_2 + (u_0-c)H_2 + H_1u_1 = 0, \tag{65b}$$

from which we have

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + \left[(1+m^2) - 6m^2\,\mathrm{cd}^2\,\xi\right]u_2 = \pm\frac{3m}{k\sqrt{\beta}}A_{\mathrm{cd}}^2\,\mathrm{cd}\,\xi\,\mathrm{sd}^2\,\xi\,\mathrm{nd}^2\,\xi. \tag{66}$$

The special solution to (66) can be assumed to be

$$u_2 = B_1\,\mathrm{cd}\,\xi + B_3\,\mathrm{cd}^3\,\xi. \tag{67}$$

Substituting (67) into (66) yields

$$B_1 = \mp\frac{(1+m^2)A_{\mathrm{cd}}^2}{4m(1-m^2)^2k\sqrt{\beta}}, \quad B_3 = \pm\frac{mA_{\mathrm{cd}}^2}{2(1-m^2)^2k\sqrt{\beta}}, \tag{68}$$

i.e. the second-order solution is

$$u_2 = \mp\frac{(1+m^2)A_{\mathrm{cd}}^2}{4m(1-m^2)^2k\sqrt{\beta}}\mathrm{cd}\,\xi \pm \frac{mA_{\mathrm{cd}}^2}{2(1-m^2)^2k\sqrt{\beta}}\mathrm{cd}^3\,\xi, \tag{69a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{69b}$$

where u_0 and u_1 are given by (60*b*) and (62).

4.2. sd ξ *expansion and Lamé function solution*

For the zeroth-order equation (20), the ansatz solution can also be assumed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{sd}\,\xi + a_2\,\mathrm{sd}^2\,\xi, \tag{70a}$$

$$u_0 = b_0 + b_1\,\mathrm{sd}\,\xi, \tag{70b}$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (70) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0,$$
$$C_2 = -\frac{c^2}{2} + \beta k^2(1-2m^2), \tag{71a}$$

$$b_0 = c, \quad b_1 = \pm 2mk\sqrt{-\beta(1-m^2)}. \tag{71b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(1-2m^2) + 2m^2(1-m^2)k^2\beta\,\mathrm{sd}^2\,\xi, \tag{72a}$$

$$u_0 = c \pm 2mk\sqrt{-\beta(1-m^2)}\,\mathrm{sd}\,\xi. \tag{72b}$$

Substituting (72) and (71*a*) into the first-order equation (21) yields

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\xi^2} + \left[(1-2m^2) + 6m^2(1-m^2)\mathrm{sd}^2\,\xi\right]u_1 = 0. \tag{73}$$

Here it is obvious that u_1 in (73) takes the same form as y in (10). Hence we can assume that u_1 takes the following form:

$$u_1 = A_{\mathrm{sd}}L_2^{\mathrm{sd}}(\xi) = A_{\mathrm{sd}}\,\mathrm{cd}\,\xi\,\mathrm{nd}\,\xi. \tag{74}$$

Substituting (74) into (26*a*) yields

$$H_1 = \mp 2mk\sqrt{-\beta(1-m^2)}A_{\mathrm{sd}}L_3^{\mathrm{cd}}(\xi). \tag{75}$$

Hence the final first-order solution is

$$H_1 = \mp 2mk\sqrt{-\beta(1-m^2)}A_{\mathrm{sd}}L_3^{\mathrm{cd}}(\xi), \quad u_1 = A_{\mathrm{sd}}L_2^{\mathrm{sd}}(\xi), \tag{76}$$

where A_{sd} is an arbitrary constant; obviously different Lamé functions $L_3^{\mathrm{cd}}(\xi)$ and $L_2^{\mathrm{sd}}(\xi)$ exist in the same system for the shallow water system.

In order to obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (72) and the first-order solution (76) back into the second-order equation (22). We have

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}\xi^2} + \left[(1-2m^2) + 6m^2(1-m^2)\mathrm{sd}^2\,\xi\right]u_2 = \mp\frac{3m\sqrt{1-m^2}}{k\sqrt{-\beta}}A_{\mathrm{sd}}^2\,\mathrm{sd}\,\xi\,\mathrm{cd}^2\,\xi\,\mathrm{nd}^2\,\xi. \tag{77}$$

The special solution to (77) can be assumed to be

$$u_2 = B_1\,\mathrm{sd}\,\xi + B_3\,\mathrm{sd}^3\,\xi. \tag{78}$$

Substituting (78) into (77) yields

$$B_1 = \pm\frac{(2m^2-1)A_{\mathrm{sd}}^2}{4mk\sqrt{-\beta(1-m^2)}}, \quad B_3 = \mp\frac{m\sqrt{1-m^2}A_{\mathrm{sd}}^2}{2k\sqrt{-\beta}}, \tag{79}$$

i.e. the second-order solution is

$$u_2 = \pm\frac{(2m^2-1)A_{\mathrm{sd}}^2}{4mk\sqrt{-\beta(1-m^2)}}\mathrm{sd}\,\xi \mp \frac{m\sqrt{1-m^2}A_{\mathrm{sd}}^2}{2k\sqrt{-\beta}}\mathrm{sd}^3\,\xi, \tag{80a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{80b}$$

where u_0 and u_1 are given by (72*b*) and (74).

4.3. nd ξ *expansion and Lamé function solution*

For the zeroth-order equation (20), the ansatz solution can also be assumed to take the following form:

$$H_0 = a_0 + a_1\,\mathrm{nd}\,\xi + a_2\,\mathrm{nd}^2\,\xi, \tag{81a}$$

$$u_0 = b_0 + b_1\,\mathrm{nd}\,\xi, \tag{81b}$$

where the expansion coefficients a_0, a_1, a_2 and b_0, b_1 can be determined by substituting (81) into (20). Here we have

$$H_0 = cu_0 - \frac{1}{2}u_0^2 + C_2, \quad C_1 = 0, \quad C_2 = -\frac{c^2}{2} + \beta k^2(m^2-2), \tag{82a}$$

$$b_0 = c, \quad b_1 = \pm 2k\sqrt{-\beta(1-m^2)}. \tag{82b}$$

Thus the zeroth-order solution for the shallow water system (15) is

$$H_0 = \beta k^2(m^2-2) + 2k^2\beta(1-m^2)\mathrm{nd}^2\,\xi, \tag{83a}$$

$$u_0 = c \pm 2k\sqrt{-\beta(1-m^2)}\mathrm{nd}\,\xi. \tag{83b}$$

Substituting (83) and (82*a*) into the first-order equation (21) yields

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + \left[(m^2-2) + 6(1-m^2)\mathrm{nd}^2\,\xi\right]u_1 = 0. \tag{84}$$

Here it is obvious that u_1 in (84) takes the same form as y in (10). Hence we can assume that u_1 takes the following form:

$$u_1 = A_{\mathrm{nd}} L_2^{\mathrm{nd}}(\xi) = A_{\mathrm{nd}}\,\mathrm{cd}\,\xi\,\mathrm{sd}\,\xi. \tag{85}$$

Substituting (85) into (26*a*) yields

$$H_1 = \mp 2k\sqrt{-\beta(1-m^2)}A_{\mathrm{nd}} L_3^{\mathrm{cd}}(\xi). \tag{86}$$

Hence the final first-order solution is

$$H_1 = \mp 2k\sqrt{-\beta(1-m^2)}A_{\mathrm{nd}} L_3^{\mathrm{cd}}(\xi), \quad u_1 = A_{\mathrm{nd}} L_2^{\mathrm{nd}}(\xi), \tag{87}$$

where A_{nd} is an arbitrary constant; obviously different Lamé functions $L_3^{\mathrm{cd}}(\xi)$ and $L_2^{\mathrm{nd}}(\xi)$ exist in the same system for the shallow water system.

In order to obtain the second-order solution of the shallow water system, we have to substitute the zeroth-order solution (83) and the first-order solution (87) back into the second-order equation (22). We have

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} + \left[(m^2-2) + 6(1-m^2)\,\mathrm{nd}^2\xi\right]u_2 = \mp\frac{3\sqrt{1-m^2}}{k\sqrt{-\beta}}A_{\mathrm{nd}}^2\,\mathrm{nd}\,\xi\,\mathrm{cd}^2\,\xi\,\mathrm{nd}^2\,\xi. \tag{88}$$

The special solution to (88) can be assumed to be

$$u_2 = B_1\,\mathrm{nd}\,\xi + B_3\,\mathrm{nd}^3\,\xi. \tag{89}$$

Substituting (89) into (88) yields

$$B_1 = \pm\frac{(2-m^2)A_{\mathrm{nd}}^2}{4m^4k\sqrt{-\beta(1-m^2)}}, \quad B_3 = \mp\frac{\sqrt{1-m^2}A_{\mathrm{nd}}^2}{2m^4k\sqrt{-\beta}}, \tag{90}$$

i.e. the second-order solution is

$$u_2 = \pm\frac{(2-m^2)A_{\mathrm{nd}}^2}{4m^4k\sqrt{-\beta(1-m^2)}}\mathrm{nd}\,\xi \mp \frac{\sqrt{1-m^2}A_{\mathrm{nd}}^2}{2m^4k\sqrt{-\beta}}\mathrm{nd}^3\,\xi, \tag{91a}$$

$$H_2 = (c-u_0)u_2 - \tfrac{1}{2}u_1^2, \tag{91b}$$

where u_0 and u_1 are given by (83*b*) and (85).

5. Conclusion and discussion

In this paper, two kinds of Lamé function are reported and applied to solve nonlinear equations, where the shallow water system is taken as an example to illustrate the applications of Lamé functions to nonlinear equations to derive two kinds of multi-order solutions when the perturbation method is involved. The results obtained in this paper are very important for the nonlinear instability of nonlinear coherent structures of nonlinear equations. Additionally, the method and results given in this paper can be easily applied to other nonlinear systems, too.

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