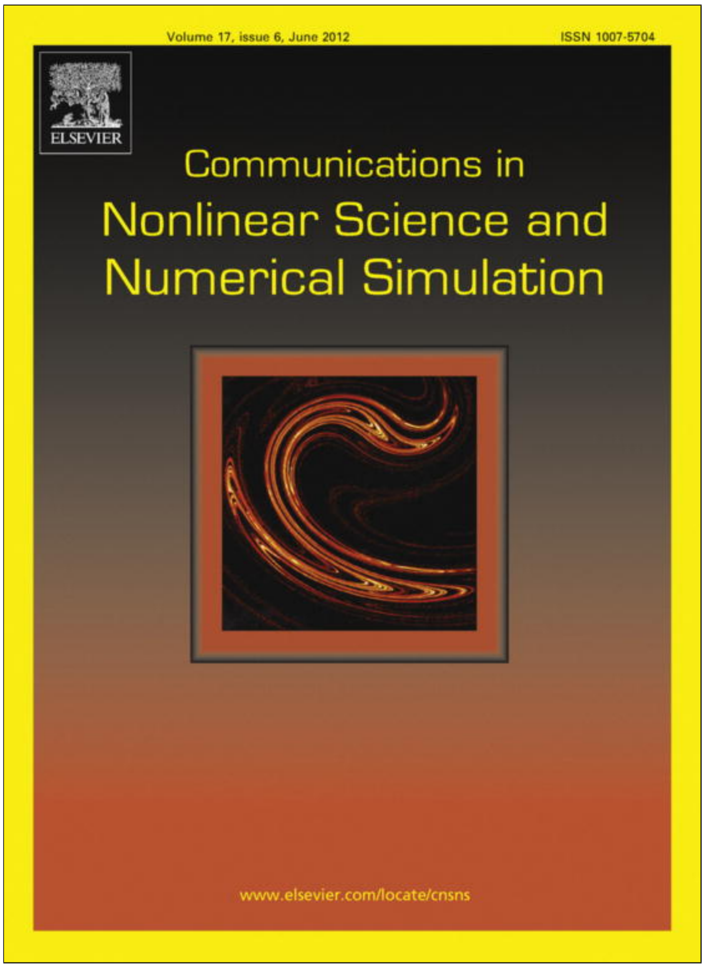
Volume 17, issue 6, June 2012
ISSN 1007-5704
ELSEVIER
Communications in
Nonlinear Science and
Numerical Simulation
www.elsevier.com/locate/cnsns

Commun Nonlinear Sci Numer Simulat 17 (2012) 2362–2371

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

# Exact coherent structures for coupled integrable dispersionless equations

Zuntao Fu [a,*], Jiangyu Mao [b]

[a] Dept. of Atmospheric and Oceanic Sciences & Laboratory for Climate and Ocean-Atmosphere Studies, School of Physics, Peking University, Beijing 100871, China
[b] LASG, Institute of Atmospheric Physics, CAS, Beijing 100029, China

ARTICLE INFO

*Article history:*
Received 28 September 2011
Received in revised form 20 October 2011
Accepted 23 October 2011
Available online 7 November 2011

*Keywords:*
Coupled integrable dispersionless equations
Singular manifold method
Coherent structure

ABSTRACT

In this paper, the singular manifold method is applied to search coherent structures in an analytical form for the coupled integrable dispersionless equations. The Generalized solutions have been derived to the coupled integrable dispersionless equations, where the solutions are determined by the singular variable totally. With the aid of symbolic computation and plot representation of Maple, some coherent structures expressed in terms of new forms, such as solitoffs and breather lattice structures, have been illustrated by means of arbitrary functions in the analytical forms.



## 1. Introduction

For the following coupled integrable dispersionless equations [1]

$$u_{xt} + (vw)_x = 0, \tag{1a}$$
$$v_{xt} - 2vu_x = 0, \tag{1b}$$
$$w_{xt} - 2wu_x = 0. \tag{1c}$$

Konno and co-workers [1] have shown that the coupled system is solvable by using the inverse scattering method. Similar property was found by Alagesan and co-workers [2] when they investigated the singularity structure analysis of these coupled system and found that these system possesses the Painlevé property by the method of singular manifold analysis [3].

Recently, Naranmandula et al. [4], Dai et al. [5], and Liu et al. [6] have obtained some exact solutions and constructed some coherent structures by the method of improved homogeneous balance, exp-function and Riccati equation mixed method and Jacobi elliptic function expansion method, respectively. The existence of these different coherent structures also tells us that there are still more new coherent structures in the (1 + 1)-dimensional systems can be found, since many methods have been proposed and widely applied to solve (1 + 1)-dimensional nonlinear wave equations extensively [7–16], and these methods can also be applied to find more coherent structures in the (1 + 1)-dimensional systems. For example, the singular manifold analysis [3] has been used widely to analyze the integrability of nonlinear systems, (1 + 1)-dimensional, (2 + 1)-dimensional or higher dimensional. In fact, this method has been extended by Peng and his co-workers [17–20] to construct the localized solutions, such as dromions [21,22] and solitoffs [23] in the (2 + 1) or higher dimensional nonlinear systems and it is shown that the singular manifold method is powerful in this direction. In this paper, we will take the coupled integrable dispersionless Eqs. (1) as an example to show there are more coherent structures in the (1 + 1)-dimensional systems by applying the singular manifold method [3] in details.

* Corresponding author. Tel.: +86 010 62767184; fax: +86 010 62751094.
*E-mail address:* fuzt@pku.edu.cn (Z. Fu).

1007-5704/$ - see front matter 
doi:10.1016/j.cnsns.2011.10.024

## 2. The analytical solutions to the coupled integrable dispersionless equations

According to the singular manifold method [3], the solutions to the coupled integrable dispersionless Eqs. (1) can be truncated as

$$u = \phi^{-1}u_0 + u_1, \tag{2a}$$
$$v = \phi^{-1}v_0 + v_1, \tag{2b}$$
$$w = \phi^{-1}w_0 + w_1, \tag{2c}$$

where $\phi = \phi(x,t)$ is the singular manifold variable, $u_i = u_i(x,t)$, $v_i = v_i(x,t)$, and $w_i = w_i(x,t)$, $i = 0,1$.

Substituting Eq. (2) into Eq. (1) and equating the coefficients with the same powers of $\phi$, one gets

$$u_0 = -\phi_t, \tag{3a}$$
$$v_0 w_0 = -\phi_t^2 \tag{3b}$$

and $u_1$, $v_1$ and $w_1$ satisfy the Eq. (1), similar results have been found by Alagesan and Porsezian [2].

Since many studies [4,6] have found that $v$ is proportional to $w$, here we take

$$v_0 = a\phi_t, \quad w_0 = -\frac{1}{a}\phi_t, \tag{4}$$

where $a$ is a non-zero constant.

Substituting (3) and (4) back into other equations for coefficients with the same powers of $\phi$, one gets

$$u_1 = \frac{\phi_{tt}}{2\phi_t} + h(t), \tag{5a}$$
$$v_1 = -\frac{a\phi_{tt}}{2\phi_t}, \tag{5b}$$
$$w_1 = \frac{\phi_{tt}}{2a\phi_t}, \tag{5c}$$

where $h(t)$ is an arbitrary function and $\phi$ satisfies the following equation

$$\left(\frac{\phi_{tt}}{\phi_t}\right)_{xt} - \left(\frac{\phi_{tt}}{\phi_t}\right)\left(\frac{\phi_{tt}}{\phi_t}\right)_x = 0, \tag{6}$$

i.e.

$$\left(\frac{\phi_{tt}}{\phi_t}\right)_t = \frac{1}{2}\left(\frac{\phi_{tt}}{\phi_t}\right)^2 + g(t), \tag{7}$$

where $g(t)$ is another arbitrary function.

If $\phi$ satisfies (6) or (7), then the solution to Eq. (1) can be written as

$$u = \left(-\frac{\phi_t}{\phi} + \frac{\phi_{tt}}{2\phi_t}\right) + h(t), \tag{8a}$$
$$v = -a\left(-\frac{\phi_t}{\phi} + \frac{\phi_{tt}}{2\phi_t}\right), \tag{8b}$$
$$w = \frac{1}{a}\left(-\frac{\phi_t}{\phi} + \frac{\phi_{tt}}{2\phi_t}\right). \tag{8c}$$

As mentioned in Ref. [3], if the arbitrary function $\phi$ takes a separable form, then (6) can be solved. For any given $\phi$, we can derive (8), generalized analytical solutions for Eq. (1). For example, if $\phi$ takes the following separable form

$$\phi = R(t) + f(x), \tag{9}$$

where $R(t)$ and $f(x)$ are two arbitrary functions. It is easy to check that $\phi$ satisfies (6), and then the solution to Eq. (1) can be written as

$$u = -\frac{R_t}{R(t)+f(x)} + \frac{R_{tt}}{2R_t} + h(t), \tag{10a}$$
$$v = -a\left[-\frac{R_t}{R(t)+f(x)} + \frac{R_{tt}}{2R_t}\right], \tag{10b}$$
$$w = \frac{1}{a}\left[-\frac{R_t}{R(t)+f(x)} + \frac{R_{tt}}{2R_t}\right]. \tag{10c}$$

Another example, if $\phi$ takes the following separable form

$$\phi = f(x)[R(t) + q(x)], \tag{11}$$

where $R(t)$, $f(x)$ and $q(x)$ are three arbitrary functions. It is easy to check that $\phi$ satisfies (6), and then the solution to Eq. (1) can be written as the same result as (10), which is totally determined by the singular variable $\phi$.

In the following part, we will show that different choices of arbitrary functions of $R(t)$ and $f(x)$ for separable $\phi$ will result in more coherent structures.

## 3. Different coherent structures

In the last section, we can see that the solution to Eq. (1) is totally determined by the singular variable $\phi$, whose two arbitrary functions $R(t)$ and $f(x)$ will be mainly in charge of the coherent structures of Eq. (1).

Thanks to the arbitrariness of functions $R(t)$ and $f(x)$, we may obtain a diversity of exact solutions to Eq. (1) by choosing these functions. For brevity, we only show the results for $u$.

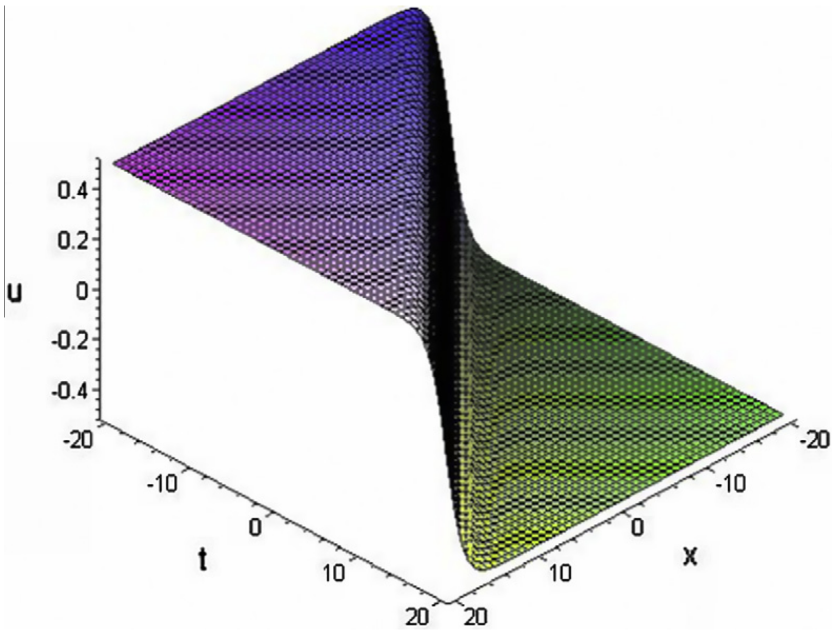


**Fig. 1.** Spatiotemporal evolution of a typical shock wave of Eq. (12).

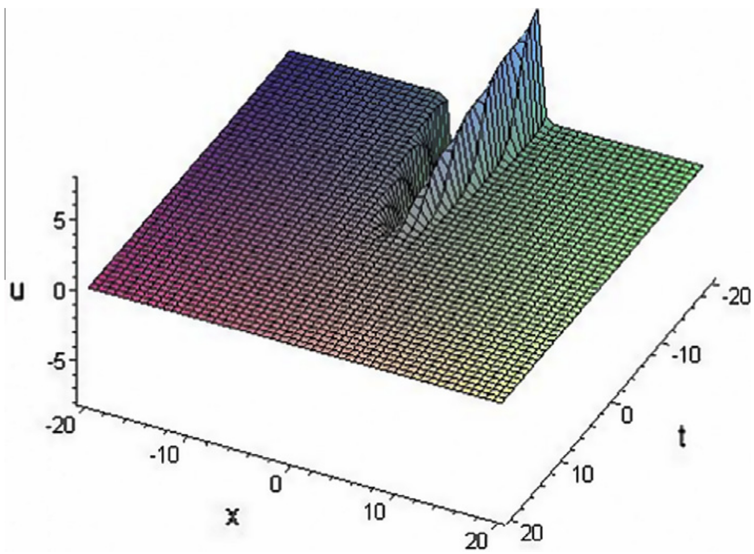


**Fig. 2.** The spatiotemporal evolution of field $u$ in Eq. (13) takes the pattern of solitoff and anti-solitoff with harmonic motions.

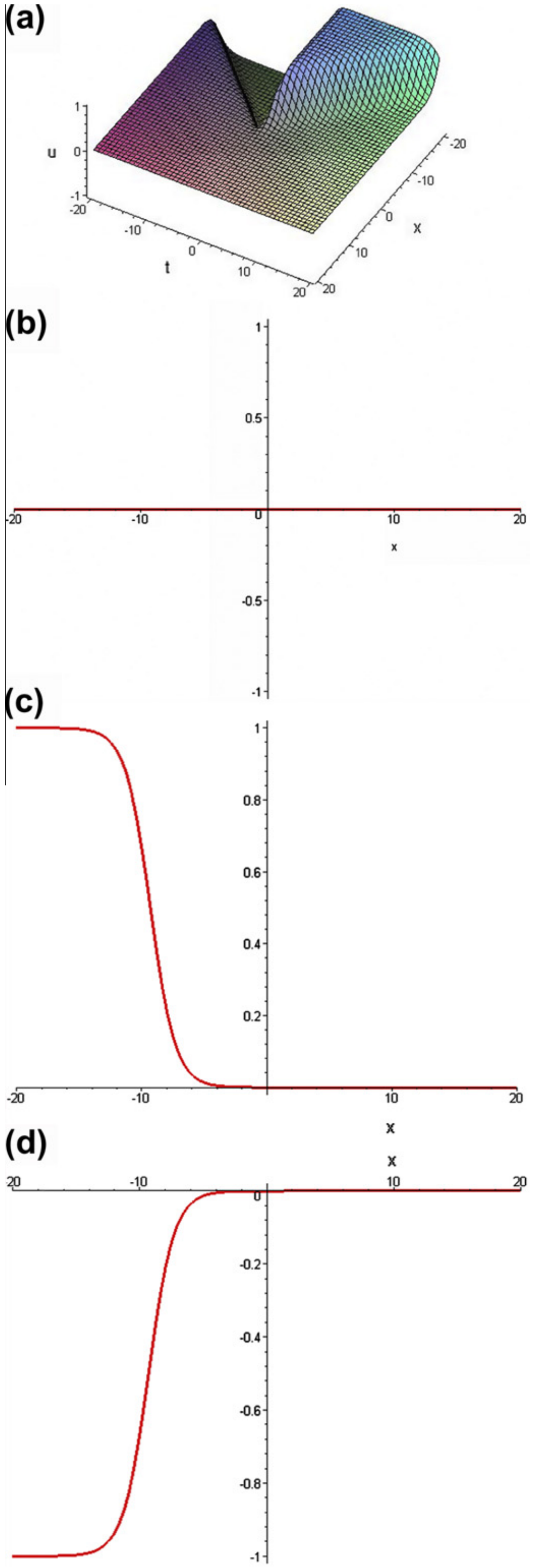


**Fig. 3.** Spatiotemporal evolution plot of kink wave and anti-kink wave reflection for Eq. (14) (a) and the snapshots at time (b) $t = 0$ (c) $t = 10$ (d) $t = -10$.

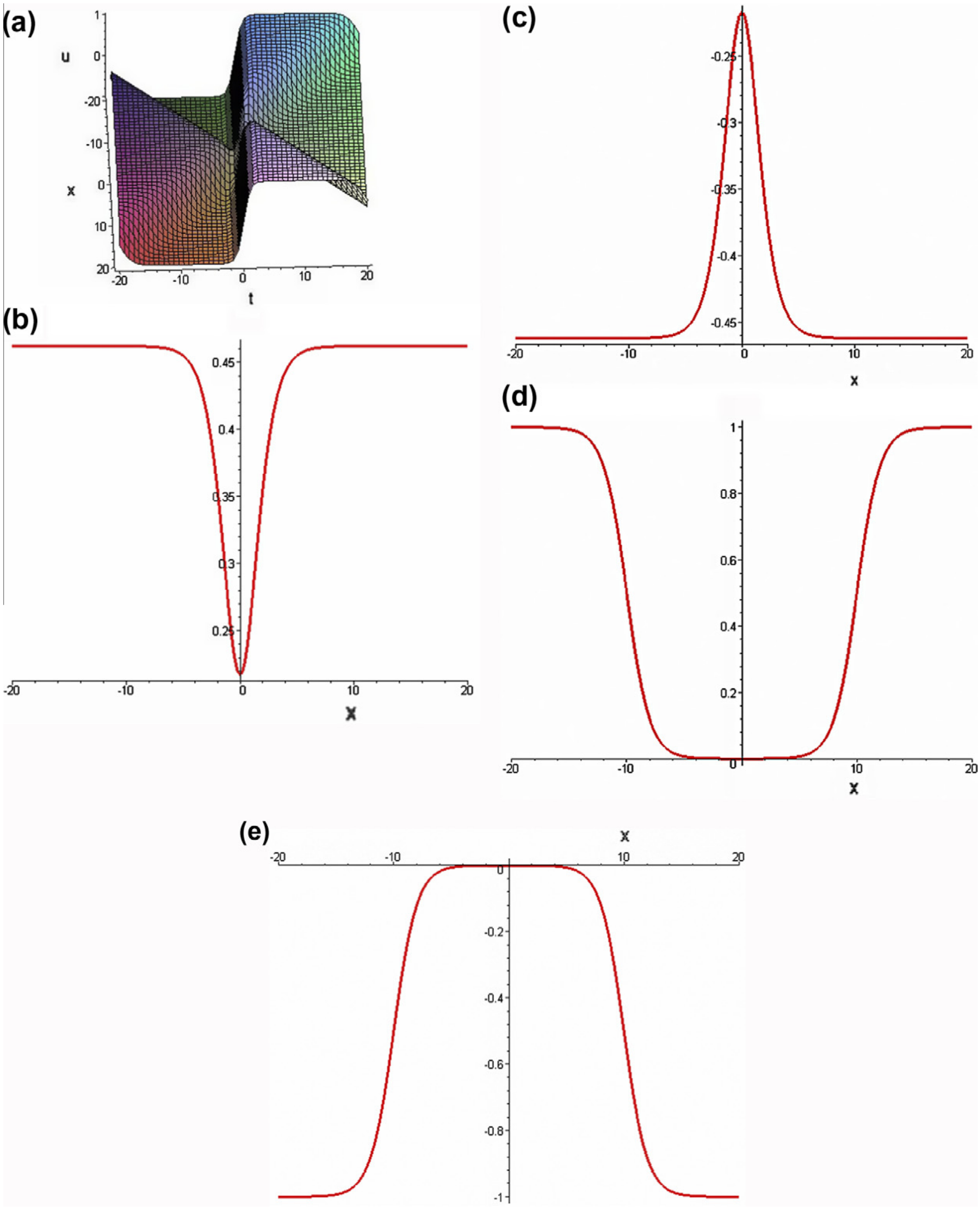


**Fig. 4.** Spatiotemporal evolution plot of kink and anti-kink interaction for Eq. (15) (a) and the snapshots at time (b) $t = 0.5$ (c) $t = -0.5$ (d) $t = 10$ (e) $t = -10$.

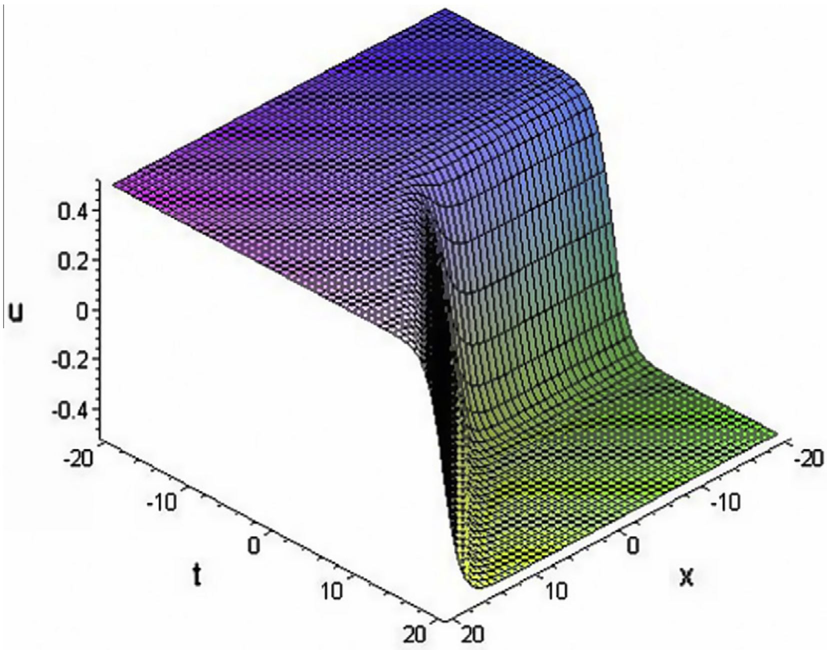


**Fig. 5.** Spatiotemporal evolution plot of vanishing shock wave type soliton for Eq. (16).

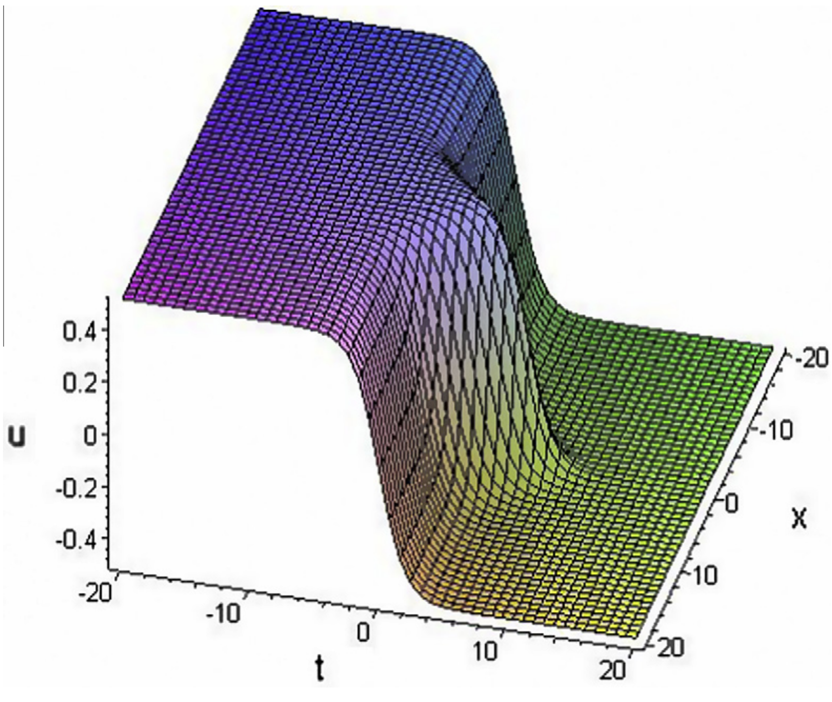


**Fig. 6.** Spatiotemporal evolution plot of flat-top soliton for Eq. (17).

**Case 1:** If $f(x) = e^x$, $R(t) = e^t$ and $h(t) = 0$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = -\frac{e^t}{e^t + e^x} + \frac{1}{2}. \tag{12}$$

In this case, field $u$ is a typical shock wave, its spatiotemporal evolution is shown in Fig. 1.

**Case 2:** If $f(x) = \operatorname{sech}(x), R(t) = e^{-t^2}$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = \frac{2te^{-t^2}}{e^{-t^2} + \operatorname{sech}(x)}. \tag{13}$$

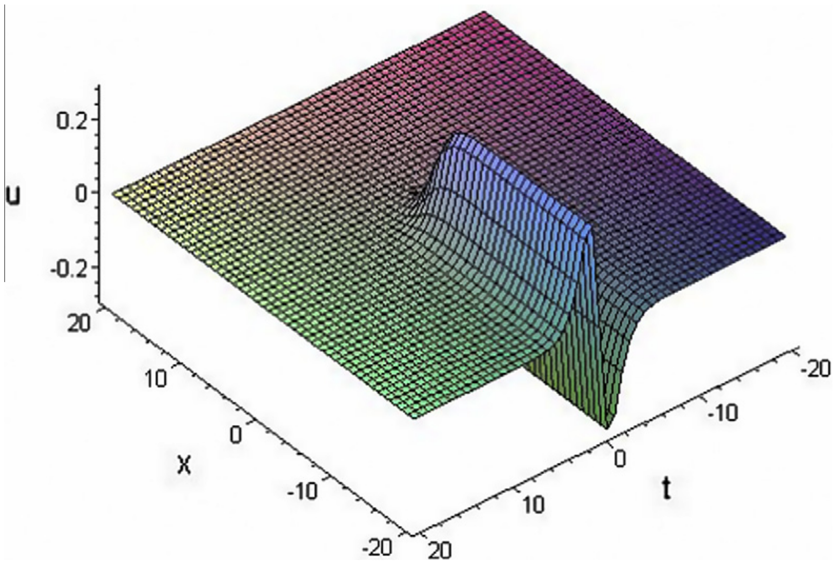


**Fig. 7.** The spatiotemporal evolution of field $u$ in Eq. (18) takes the pattern of solitoff and anti-solitoff.

In this case, the spatiotemporal evolution of field $u$ takes the pattern of solitoff and anti-solitoff with harmonic motions, which is illustrated in Fig. 2.

**Case 3:** If

$f(x) = e^x$, $R(t) = \text{sech}\,(t)$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = \frac{\text{sech}\,(t)\tanh(t)}{\text{sech}\,(t) + e^x}. \tag{14}$$

In this case, field $u$ takes the reflection coherent structure, its spatiotemporal evolution plot of kink wave and anti-kink wave reflection is given in Fig. 3, where a traveling kink wave reflects at $x = 0$ and then becomes an anti-kink wave.

**Case 4:** If

$f(x) = \text{sech}(x)$, $R(t) = \text{sech}(t)$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = \frac{\text{sech}\,(t)\tanh(t)}{\text{sech}\,(t) + \text{sech}\,x}. \tag{15}$$

In this case, field $u$ takes the interaction coherent structure, its spatiotemporal evolution plot of kink wave and anti-kink wave interaction is depicted in Fig. 4, where a traveling kink wave heads on and collides with anti-kink wave.

**Case 5:** If

$f(x) = 1 + e^x$, $R(t) = e^t$ and $h(t) = 0$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = -\frac{e^t}{1 + e^t + e^x} + \frac{1}{2}. \tag{16}$$

In this case, field $u$ is a typical vanishing shock wave, its spatiotemporal evolution is shown in Fig. 5.

**Case 6:** If

$f(x) = 1 + 500\,\text{sech}(x)$, $R(t) = e^t$ and $h(t) = 0$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = -\frac{e^t}{1 + e^t + 500\text{sech}\,(x)} + \frac{1}{2}. \tag{17}$$

In this case, field $u$ takes flat-top soliton pattern, its spatiotemporal evolution is shown in Fig. 6.

**Case 7:** If

$f(x) = 1 + e^x$, $R(t) = \text{sech}(t)$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = \frac{\text{sech}\,(t)\tanh(t)}{1 + \text{sech}\,(t) + e^x}. \tag{18}$$

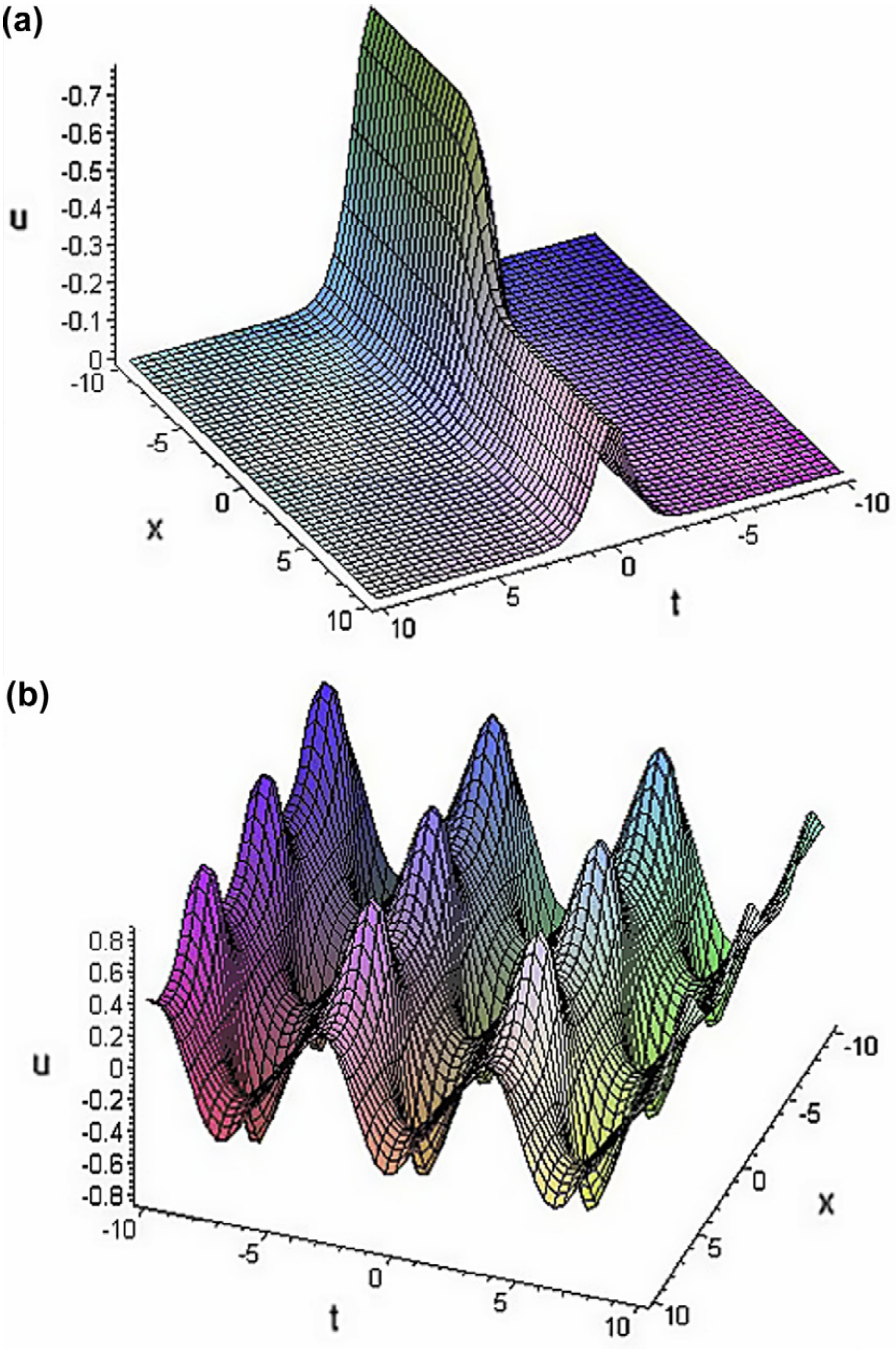


**Fig. 8.** The spatiotemporal evolution of field $u$ in Eq. (19) takes the pattern of solitoff and line soliton interaction at $m = 1$, (a) and breather lattice structure at $m = 0.5$ (b).

In this case, the spatiotemporal evolution of field $u$ takes the pattern of solitoff and anti-solitoff, which is illustrated in Fig. 7.
**Case 8:** If

$f(x) = 2.5 + \mathrm{sn}(x,m)$, $R(t) = \mathrm{sn}(t,m)$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = -\frac{\mathrm{cn}(t,m)\mathrm{dn}(t,m)}{2.5 + \mathrm{sn}(t,m) + \mathrm{sn}(x,m)}, \tag{19}$$

where $\mathrm{sn}(y,m)$, $\mathrm{cn}(y,m)$ and $\mathrm{dn}(y,m)$ are the Jacobi elliptic sine function, the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind with its modulus $m$ $(0 < m < 1)$ [24,25], respectively.
In this case, the spatiotemporal evolution of field $u$ takes the pattern of solitoff and line soliton interaction at $m = 1$, which is illustrated in Fig. 8a and breather lattice structure at $m = 0.5$ [26,27], which is illustrated in Fig. 8b.

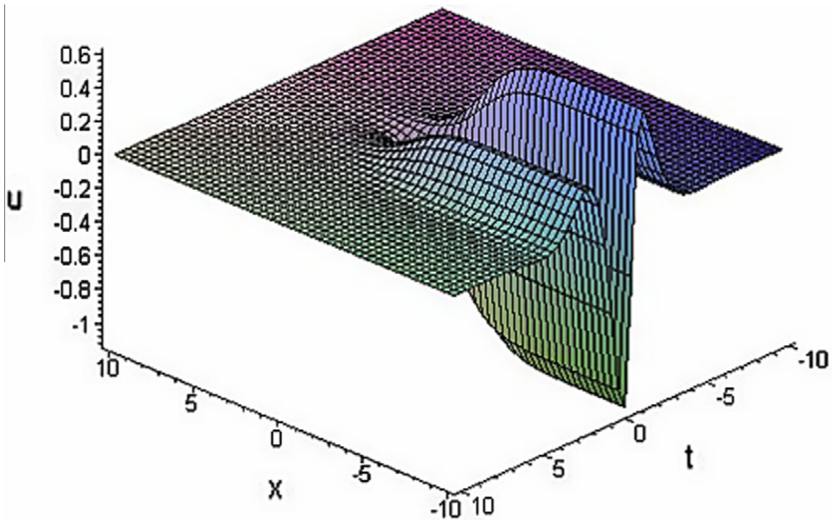


**Fig. 9.** The spatiotemporal evolution of field $u$ in Eq. (20) takes the pattern of anti-solitoff, solitoff and anti-solitoff.

**Case 9:** If $f(x) = 1 + e^x, R(t) = \frac{\sin(t)}{\cosh(t)}$ and $h(t) = -\frac{R_{tt}}{2R_t}$, then the solution to $u$ in Eq. (1) can be expressed as

$$u = \frac{\cos(t)\cosh(t) - \sin(t)\sinh(t)}{\sin(t)\cosh(t) + (1 + e^x)\cosh(t)^2}. \tag{20}$$

In this case, the spatiotemporal evolution of field $u$ takes the pattern of anti-solitoff, solitoff and anti-solitoff, which is illustrated in Fig. 9.

## 4. Conclusion and discussion

In this paper, the singular manifold method is applied to the (1 + 1)-dimensional nonlinear systems, certain special coherent structures have been obtained because of the existence of arbitrary functions in singular variable $\phi$. For the coupled integrable dispersionless equations, its solutions are totally determined by the singular variable $\phi$, which has two or three arbitrary functions. When these arbitrary functions take different analytical forms, more different coherent structures can be presented, some of them are new and unreported in the literature. So more applications of this method to other nonlinear systems, (1 + 1)-dimensional, (2 + 1)-dimensional or higher dimensional nonlinear equations to derive more new structures deserves to be studied further.

## Acknowledgement

Many thanks are due to support from National Natural Science Foundation of China (No. 40975027).

## References

[1] Konno K, Oono H. J Phys Soc Jpn 1994;63:377.
[2] Alagesan T, Porsezian K. Chaos Solitons Fract 1997;8:1645.
[3] Weiss J, Tabor M. G Carnevale J Math Phys 1983;24:522.
[4] Naranmendula, Han YC, Mandafu. Commun Theor Phys 2009;51:1037.
[5] Dai CQ, Yang Q, Wang YY. Commun Theor Phys 2011;55:622.
[6] Liu SK, Zhao Q, Liu SD. Chin Phys Lett B 2011;20:040202.
[7] Kudryashov NA. Phys Lett A 1990;147:287–91.
[8] Yan CT. Phys Lett A 1996;224:77–84.
[9] Liu SK, Fu ZT, Liu SD, Zhao Q. Phys Lett A 2001;289:69–74.
[10] Liu SK, Fu ZT, Liu SD, Zhao Q. Appl Math Mech 2001;22:326–31.
[11] Fu ZT, Liu SK, Liu SD, Zhao Q. Phys Lett A 2001;290:72–6.
[12] Dou FQ, Sun JA, Duan WS, et al. Commun Theor Phys 2006;45:1063.
[13] Wazwaz AM. Physica D 2006;213:147.
[14] He JH, Wu XH. Chaos Solitons Fract 2006;30:700–6.
[15] Liu XP, Liu CP. Chaos Solitons Fract 2009;39:1915–9.
[16] Liu GT. Appl Math Comput 2009;212:312.
[17] Peng YZ. Commun Theor Phys 2004;41:669.
[18] Peng YZ. Phys Lett A 2006;351:41.
[19] Peng YZ. Rep Math Phys 2007;60:259.
[20] Peng YZ, Feng H, Krishnan EV. Appl Math Model 2009;33:1842.
[21] Fokas A, Santini PM. Phys Rev Lett 1989;63:1329.

[22] Hietarinta J, Hirota R. Phys Lett A 1990;145:237.
[23] Gilson CR. Phys Lett A 1992;161:423.
[24] Wang ZX, Guo DR. Special functions. Singapore: World Scientific; 1989.
[25] Liu SK, Liu SD. Nonlinear equations in physics. Beijing: Peking University Press; 2000.
[26] Fu ZT, Liu SD, Liu SK. J Phys A 2007;40:4739.
[27] Fu ZT, Liu SD, Liu SK. Phys Scr 2007;76:15.