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Applications of Elliptic Equation to Nonlinear Coupled Systems*

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Abstract *The elliptic equation is taken as a transformation and applied to solve nonlinear coupled systems. It is shown that this method is more powerful to give more kinds of solutions, such as rational solutions, solitary wave solutions, periodic wave solutions and so on, so this method can be taken as a unified method in solving nonlinear coupled systems.*

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1 Introduction

Since more and more problems have to involve nonlinearity, how to solve these nonlinear models attracts much attention, and many methods have been proposed to construct exact solutions to nonlinear equations. Among them are the function transformation method,[1−4] the homogeneous balance method,[5−7] the hyperbolic function expansion method,[8−11] the Jacobi elliptic function expansion method,[12,13] the nonlinear transformation method,[14,15] the trial function method[16,17] and others.[18−20]

Among these methods, a transformation is often introduced to simplify solving processes.[1−4,8,10] For example, in the hyperbolic function expansion methods, when a transformation is introduced, one can get more kinds of solutions. For example, Fan[8] introduced

$$u = u(w)\,, \quad \frac{\mathrm{d}w}{\mathrm{d}\xi} = b + w^2\,, \quad w = w(\xi)\,, \quad \xi = x - ct\,, \tag{1}$$

and Yan *et al.*[10] extended it as

$$u = u(w)\,, \qquad \frac{\mathrm{d}w}{\mathrm{d}\xi} = R(1 + \mu w^2)\,. \tag{2}$$

Actually, there Fan and Yan applied the well-known Riccati equation as their transformations.

In this paper, we will consider elliptic equation[21]

$$y'^2 = \sum_{i=0}^{4} a_i y^i\,, \qquad a_4 \neq 0\,, \tag{3}$$

where $y' = \mathrm{d}y/\mathrm{d}\xi$, and take it as a new transformation to solve nonlinear wave equations. In the following sections, applications of transformation (3) to some well-known nonlinear coupled systems will be discussed.

2 Coupled mKdV Equations

We here consider coupled mKdV equations of the following form

$$u_t + \alpha u^2 u_x + \beta u_{xxx} + c_0 v_x = 0\,,$$
$$v_t + \gamma v v_x + \delta(uv)_x = 0\,. \tag{4}$$

Seeking their solution in the following frame

$$u = u(\xi)\,, \quad v = v(\xi)\,, \qquad \xi = x - ct\,, \tag{5}$$

then we can get

$$-cu' + \alpha u^2 u' + \beta u''' + c_0 v' = 0\,,$$
$$-cv' + \gamma v v' + \delta(uv)' = 0\,. \tag{6}$$

And then we suppose that equations (4) have the following solution

$$u = u(y) = \sum_{j_1=0}^{n_1} b_{j_1} y^{j_1}\,, \quad v = v(y) = \sum_{j_1=0}^{n_2} d_{j_2} y^{j_2}\,, \tag{7}$$

where y satisfies the elliptic equation (3), then

$$y'' = \frac{a_1}{2} + a_2 y + \frac{3a_3}{2} y^2 + 2a_4 y^3\,,$$
$$y''' = (a_2 + 3a_3 y + 6a_4 y^2) y'\,. \tag{8}$$

There n in Eq. (7) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eqs. (4). Here we know that the degrees of u and v are

$$O(u) = O(y^{n_1}) = n_1\,, \quad O(v) = O(y^{n_2}) = n_2\,, \tag{9}$$

and from Eqs. (3) and (8), one has

$$O(y'^2) = O(y^4) = 4\,, \quad O(y'') = O(y^3) = 3\,. \tag{10}$$

Actually one can get

$$O(y^{(l)}) = l + 1\,. \tag{11}$$

So one has

$$O(u) = n_1\,, \quad O(v) = n_2\,, \quad O(u') = n_1 + 1\,,$$
$$O(u'') = n_1 + 2\,, \qquad O(u^{(l)}) = n_1 + l\,. \tag{12}$$

For coupled mKdV equations (4), we have $n_1 = 1$ and $n_2 = 1$, so the ansatz solution (7) can be rewritten as

$$u = b_0 + b_1 y\,, \quad v = d_0 + d_1 y\,, \quad b_1 \neq 0\,, \quad d_1 \neq 0\,. \tag{13}$$

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Substituting Eqs. (13) into Eqs. (6) leads to

$$b_0 = \mp\frac{3\beta a_3}{2\alpha}\sqrt{-\frac{\alpha}{6\beta a_4}}\,, \quad b_1 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}\,,$$

$$d_1 = \mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}\,,$$

$$d_0 = -\frac{3\beta a_3^2}{4\gamma a_4} + \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \pm \frac{3\delta\beta a_3}{\gamma\alpha}\sqrt{-\frac{\alpha}{6\beta a_4}}\,. \quad (14)$$

So if $a_3 = 0$, then

$$b_0 = 0\,, \qquad b_1 = \pm\sqrt{-\frac{6\beta a_4}{\alpha}}\,,$$

$$d_1 = \mp\frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_4}{\alpha}}\,, \quad d_0 = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2}\,, \quad (15)$$

and if we take the arbitrary constant $a_1 = 0$ then the transformation (3) takes the following form

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4\,, \quad (16)$$

which has many more kinds of solutions, some of which we will show next.

Case A Consider $a_0 = 0$, then

$$y'^2 = a_2 y^2 + a_4 y^4\,, \quad (17)$$

and we have two kinds of solutions:

(i) If $a_2 > 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{\frac{a_2}{a_4}}\,\operatorname{csch}(\sqrt{a_2}\,\xi)\,, \quad (18)$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta a_2}{\alpha}}\,\operatorname{csch}(\sqrt{a_2}\,\xi) \quad (19)$$

$$v = d_0 + d_1 y = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta a_2}{\alpha}}\,\operatorname{csch}(\sqrt{a_2}\,\xi)\,. \quad (20)$$

(ii) If $a_2 < 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{-\frac{a_2}{a_4}}\,\csc(\sqrt{-a_2}\,\xi)\,, \quad (21)$$

and

$$u = b_1 y = \pm\sqrt{\frac{6\beta a_2}{\alpha}}\,\csc(\sqrt{-a_2}\,\xi)\,, \quad (22)$$

$$v = d_0 + d_1 y = \frac{2\beta a_2}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{\frac{6\beta a_2}{\alpha}}\,\csc(\sqrt{-a_2}\,\xi)\,. \quad (23)$$

These two kinds of solutions deal with "hot spots" or "blow-up" of solutions,[22−25] which can develop singularity at a finite point.

Case B Consider $a_0 = a_2 = 0$ and $a_4 > 0$, so $b_0 = 0$, then

$$y = \pm\frac{1}{\sqrt{a_4}\xi}\,, \quad (24)$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\frac{1}{\xi}\,, \quad (25)$$

$$v = d_0 + d_1 y = -\frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,\frac{1}{\xi}\,, \quad (26)$$

which are rational solutions. The rational solutions are a disjoint union of manifolds and the particle system describing the motion of pole of rational solutions, which have been discussed in many literatures, such as Refs. [26] and [27].

Case C Consider transformation (16) directly, from which many more solutions expressed in terms of different elliptic functions[21] can be got.

(i) If $a_0 = 1$, $a_2 = -(1+m^2)$, and $a_4 = m^2$ (where $0 \le m \le 1$, m is called modulus of Jacobi elliptic functions, see Refs. [21] and [28] ∼ [30]), then the solution is

$$y = \operatorname{sn}(\xi, m)\,, \quad (27)$$

where $\operatorname{sn}(\xi, m)$ is Jacobi elliptic sine function (see Refs. [21] and [28] ∼ [30]), and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}m\,\operatorname{sn}(\xi, m)\,, \quad (28)$$

$$v = d_0 + d_1 y = -\frac{2\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}m\,\operatorname{sn}(\xi, m)\,. \quad (29)$$

(ii) If $a_0 = m^2$, $a_2 = -(1+m^2)$, and $a_4 = 1$, then the solution is

$$y = \operatorname{ns}(\xi, m) \equiv \frac{1}{\operatorname{sn}(\xi, m)}\,, \quad (30)$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\operatorname{ns}(\xi, m)\,, \quad (31)$$

$$v = d_0 + d_1 y = -\frac{2\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,\operatorname{ns}(\xi, m)\,. \quad (32)$$

(iii) If $a_0 = -m^2$, $a_2 = 2m^2 - 1$, and $a_4 = 1 - m^2$, then the solution is

$$y = \operatorname{nc}(\xi, m) \equiv \frac{1}{\operatorname{cn}(\xi, m)}\,, \quad (33)$$

where $\operatorname{cn}(\xi, m)$ is Jacobi elliptic cosine function (see Refs. [21] and [28] ∼ [30]) and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta(1-m^2)}{\alpha}}\,\operatorname{nc}(\xi, m)\,, \quad (34)$$

$$v = d_0 + d_1 y = \frac{2\beta(2m^2-1)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta(1-m^2)}{\alpha}}\operatorname{nc}(\xi, m)\,. \quad (35)$$

(iv) If $a_0 = 1$, $a_2 = 2 - m^2$, and $a_4 = 1 - m^2$, then the solution is

$$y = \mathrm{sc}\,(\xi, m) \equiv \frac{\mathrm{sn}\,(\xi, m)}{\mathrm{cn}\,(\xi, m)}\,, \tag{36}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta(1-m^2)}{\alpha}}\,\mathrm{sc}\,(\xi, m)\,, \tag{37}$$

$$v = d_0 + d_1 y = \frac{2\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta(1-m^2)}{\alpha}}\,\mathrm{sc}\,(\xi, m)\,. \tag{38}$$

(v) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, and $a_4 = 1$, then the solution is

$$y = \mathrm{cs}\,(\xi, m) \equiv \frac{\mathrm{cn}\,(\xi, m)}{\mathrm{sn}\,(\xi, m)}\,, \tag{39}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{cs}\,(\xi, m)\,, \tag{40}$$

$$v = d_0 + d_1 y = \frac{2\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{cs}\,(\xi, m)\,. \tag{41}$$

(vi) If $a_0 = 1$, $a_2 = -(1 + m^2)$, and $a_4 = m^2$, then the solution is

$$y = \mathrm{cd}\,(\xi, m) \equiv \frac{\mathrm{cn}\,(\xi, m)}{\mathrm{dn}\,(\xi, m)}\,, \tag{42}$$

where $\mathrm{dn}\,(\xi, m)$ is Jacobi elliptic function of the third kind (see Refs. [21] and [28] $\sim$ [30]) and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}m\,\mathrm{cd}\,(\xi, m)\,, \tag{43}$$

$$v = d_0 + d_1 y = -\frac{2\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}m\,\mathrm{cd}\,(\xi, m)\,. \tag{44}$$

(vii) If $a_0 = m^2(m^2 - 1)$, $a_2 = 2m^2 - 1$, and $a_4 = 1$, then the solution is

$$y = \mathrm{ds}\,(\xi, m) \equiv \frac{\mathrm{dn}\,(\xi, m)}{\mathrm{sn}\,(\xi, m)}\,, \tag{45}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{ds}\,(\xi, m)\,, \tag{46}$$

$$v = d_0 + d_1 y = \frac{2\beta(2m^2-1)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{ds}\,(\xi, m)\,. \tag{47}$$

(viii) If $a_0 = m^2$, $a_2 = -(1 + m^2)$, and $a_4 = 1$, then the solution is

$$y = \mathrm{dc}\,(\xi, m) \equiv \frac{\mathrm{dn}\,(\xi, m)}{\mathrm{cn}\,(\xi, m)}\,, \tag{48}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{dc}\,(\xi, m)\,, \tag{49}$$

$$v = d_0 + d_1 y = -\frac{2\beta(1+m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta}{\alpha}}\,\mathrm{dc}\,(\xi, m)\,. \tag{50}$$

(ix) If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, and $a_4 = -m^2$, then the solution is

$$y = \mathrm{cn}\,(\xi, m)\,, \tag{51}$$

and

$$u = b_1 y = \pm\sqrt{\frac{6\beta}{\alpha}}m\,\mathrm{cn}\,(\xi, m)\,, \tag{52}$$

$$v = d_0 + d_1 y = \frac{2\beta(2m^2-1)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{\frac{6\beta}{\alpha}}\,\mathrm{cn}\,(\xi, m)\,. \tag{53}$$

(x) If $a_0 = 1 - m^2$, $a_2 = 2 - m^2$, and $a_4 = -1$, then the solution is

$$y = \mathrm{dn}\,(\xi, m)\,, \tag{54}$$

and

$$u = b_1 y = \pm\sqrt{\frac{6\beta}{\alpha}}\,\mathrm{dn}\,(\xi, m)\,, \tag{55}$$

$$v = d_0 + d_1 y = \frac{2\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{\frac{6\beta}{\alpha}}\,\mathrm{dn}\,(\xi, m)\,. \tag{56}$$

(xi) If $a_0 = -1$, $a_2 = 2 - m^2$, and $a_4 = m^2 - 1$, then the solution is

$$y = \mathrm{nd}\,(\xi, m) \equiv \frac{1}{\mathrm{dn}\,(\xi, m)}\,, \tag{57}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta(m^2-1)}{\alpha}}\,\mathrm{nd}\,(\xi, m)\,, \tag{58}$$

$$v = d_0 + d_1 y = \frac{2\beta(2-m^2)}{\gamma} - \frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta(m^2-1)}{\alpha}}\,\mathrm{nd}\,(\xi, m)\,. \tag{59}$$

(xii) If $a_0 = 1$, $a_2 = 2m^2 - 1$, and $a_4 = (m^2 - 1)m^2$, then the solution is

$$y = \mathrm{sd}\,(\xi, m) \equiv \frac{\mathrm{sn}\,(\xi, m)}{\mathrm{dn}\,(\xi, m)}\,, \tag{60}$$

and

$$u = b_1 y = \pm\sqrt{-\frac{6\beta(m^2-1)m^2}{\alpha}}\,\mathrm{sd}\,(\xi, m)\,, \tag{61}$$

$$v = d_0 + d_1 y = \frac{2\beta(2m^2-1)}{\gamma}$$

$$-\frac{4\delta c_0}{\gamma^2} \mp \frac{2\delta}{\gamma}\sqrt{-\frac{6\beta(m^2-1)m^2}{\alpha}}\,\mathrm{sd}\,(\xi, m)\,. \tag{62}$$

And among the Jacobi elliptic functions, sine Jacobi elliptic function, cosine Jacobi elliptic functions and Jacobi elliptic function of the third kind are three basic ones, and all other Jacobi elliptic functions can be expressed in terms of them. These periodic solutions expressed in terms of each Jacobi elliptic function or some Jacobi elliptic functions have their physical meanings respectively, especially, it is known that when $m \to 1$, $\mathrm{sn}\,(\xi, m) \to \tanh\xi$, $\mathrm{cn}\,(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}\,(\xi, m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}\,(\xi, m) \to \sin\xi$, $\mathrm{cn}\,(\xi, m) \to \cos\xi$. So we can also get more kinds of solutions expressed in terms of hyperbolic functions and trigonometric functions, i.e. we can get more solitary wave solutions, kink solutions, singular wave solutions, and so on.

3 System of Variant Boussinesq Equations

The system of variant Boussinesq equations reads[5]

$$H_t + (Hu)_x + u_{xxx} = 0\,, \quad u_t + H_x + uu_x = 0\,, \tag{63}$$

which is a model for water waves, where $u(x,t)$ is the velocity and $H(x,t)$ is the total depth.

We seek its travelling wave solutions in the following frame,

$$u = u(\xi)\,, \quad H = H(\xi)\,, \quad \xi = x - ct\,, \tag{64}$$

where c is wave velocity.

And then we suppose that equations (63) have the following solution

$$H = H(y) = \sum_{j_1=0}^{n_1} b_{j_1} y^{j_1}\,, \quad u = u(y) = \sum_{j_2=0}^{n_2} d_{j_2} y^{j_2}\,, \tag{65}$$

where y satisfies the elliptic equation (3). There n in Eq. (65) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eqs. (63). For the system of variant Boussinesq equations (63), we have $n_1 = 2$ and $n_2 = 1$, so the ansatz solution (65) can be rewritten as

$$H = b_0 + b_1 y + b_2 y^2,\ u = d_0 + d_1 y,\ b_2 \neq 0\,,\ d_1 \neq 0. \tag{66}$$

Then substituting Eq. (66) into Eq. (64) and collecting each order of y yields the algebraic equations about coefficients $b_j (j = 0,1,2)$, $d_j (j = 0,1)$, and a_i $(i = 0,1,2,3,4)$,

$$b_2 = -2a_4\,, \quad b_1 = -a_3\,, \quad b_0 = -\frac{1}{2}\Big[a_2 \pm \frac{ca_3}{2\sqrt{a_4}}\Big],$$
$$d_1 = \pm 2\sqrt{a_4}\,, \qquad d_0 = c \pm \frac{a_3}{2\sqrt{a_4}}\,. \tag{67}$$

So if $a_3 = 0$, then

$$b_2 = -2a_4\,, \quad b_1 = 0\,, \quad b_0 = -\frac{a_2}{2}\,,$$
$$d_0 = c\,, \qquad d_1 = \pm 2\sqrt{a_4}\,, \tag{68}$$

and if we let the arbitrary constant $a_1 = 0$, then the transformation (3) takes the following form

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4\,, \tag{69}$$

which has many more kinds of solutions, some of which we will show next.

Case A Consider $a_0 = 0$, then

$$y'^2 = a_2 y^2 + a_4 y^4\,, \tag{70}$$

we have two kinds of solutions.

(i) If $a_2 > 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{\frac{a_2}{a_4}}\,\mathrm{csch}\,(\sqrt{a_2}\,\xi)\,, \tag{71}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2$$
$$= -\frac{a_2}{2} - 2a_2 \mathrm{csch}^2(\sqrt{a_2}\,\xi)\,, \tag{72}$$
$$u = d_0 + d_1 y$$
$$= c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{a_2}\,\mathrm{csch}\,(\sqrt{a_2}\,\xi)\,. \tag{73}$$

(ii) If $a_2 < 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{-\frac{a_2}{a_4}}\,\csc\,(\sqrt{-a_2}\,\xi)\,, \tag{74}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2$$
$$= -\frac{a_2}{2} + 2a_2 \csc^2(\sqrt{-a_2}\,\xi)\,, \tag{75}$$
$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y$$
$$= c \pm 2\sqrt{-a_2}\,\csc\,(\sqrt{-a_2}\,\xi)\,. \tag{76}$$

Case B Consider $a_0 = a_2 = 0$ and $a_4 > 0$, so $b_0 = 0$, then

$$y = \pm\frac{1}{\sqrt{a_4}\,\xi}\,, \tag{77}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = \frac{2}{\xi^2}\,, \tag{78}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm \frac{2}{\xi}\,. \tag{79}$$

Case C Consider transformation (69) directly, from which many more solutions expressed in terms of different elliptic functions[21] can be got. Here we show some more generalized solutions.

(i) If $a_0 = \mu^2 A^2$, $a_2 = -\mu^2(1+m^2)$, and $a_4 = \mu^2 m^2/A^2$, then the solution is

$$y = A\,\mathrm{sn}\,(\mu\,\xi, m)\,, \tag{80}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = \frac{\mu^2(1+m^2)}{2} - 2\mu^2 m^2 \mathrm{sn}^2(\mu\,\xi, m)\,, \tag{81}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2 m^2}\, \mathrm{sn}\,(\mu\,\xi, m)\,. \tag{82}$$

(ii) If $a_0 = \mu^2 m^2 A^2$, $a_2 = -\mu^2(1+m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\ \mathrm{ns}\,(\mu\,\xi, m)\,, \tag{83}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = \frac{\mu^2(1+m^2)}{2} - 2\mu^2 \mathrm{ns}^{\,2}(\mu\,\xi, m)\,, \tag{84}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2}\ \mathrm{ns}\,(\mu\,\xi, m)\,. \tag{85}$$

(iii) If $a_0 = -\mu^2 m^2 A^2$, $a_2 = \mu^2(2m^2-1)$, and $a_4 = \mu^2(1-m^2)/A^2$, then the solution is

$$y = A\ \mathrm{nc}\,(\mu\,\xi, m)\,, \tag{86}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = -\frac{\mu^2(2m^2-1)}{2} - 2\mu^2(1-m^2)\,\mathrm{nc}^{\,2}(\mu\,\xi, m)\,, \tag{87}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2(1-m^2)}\ \mathrm{nc}\,(\mu\,\xi, m)\,. \tag{88}$$

(iv) If $a_0 = \mu^2 A^2$, $a_2 = \mu^2(2-m^2)$, and $a_4 = \mu^2(1-m^2)/A^2$, then the solution is

$$y = A\ \mathrm{sc}\,(\mu\,\xi, m)\,, \tag{89}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = -\frac{\mu^2(2-m^2)}{2} - 2\mu^2(1-m^2)\ \mathrm{sc}^{\,2}(\mu\,\xi, m)\,, \tag{90}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2(1-m^2)}\ \mathrm{sc}\,(\mu\,\xi, m)\,. \tag{91}$$

(v) If $a_0 = \mu^2(1-m^2)A^2$, $a_2 = \mu^2(2-m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\ \mathrm{cs}\,(\mu\,\xi, m)\,, \tag{92}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = -\frac{\mu^2(2-m^2)}{2} - 2\mu^2 \mathrm{cs}^{\,2}(\mu\,\xi, m)\,, \tag{93}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2}\ \mathrm{cs}\,(\mu\,\xi, m)\,. \tag{94}$$

(vi) If $a_0 = \mu^2 A^2$, $a_2 = -\mu^2(1+m^2)$, and $a_4 = \mu^2 m^2/A^2$, then the solution is

$$y = A\ \mathrm{cd}\,(\mu\,\xi, m)\,, \tag{95}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = \frac{\mu^2(1+m^2)}{2} - 2\mu^2 m^2 \mathrm{cd}^{\,2}(\mu\,\xi, m)\,, \tag{96}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2 m^2}\ \mathrm{cd}\,(\mu\,\xi, m)\,. \tag{97}$$

(vii) If $a_0 = \mu^2 m^2(m^2-1)A^2$, $a_2 = \mu^2(2m^2-1)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\ \mathrm{ds}\,(\mu\,\xi, m)\,, \tag{98}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = -\frac{\mu^2(2m^2-1)}{2} - 2\mu^2 \mathrm{ds}^{\,2}(\mu\,\xi, m)\,, \tag{99}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2}\ \mathrm{ds}\,(\mu\,\xi, m)\,. \tag{100}$$

(viii) If $a_0 = \mu^2 m^2 A^2$, $a_2 = -\mu^2(1+m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\ \mathrm{dc}\,(\mu\,\xi, m)\,, \tag{101}$$

and

$$H = b_0 + b_2 y^2 = -\frac{a_2}{2} - 2a_4 y^2 = \frac{\mu^2(1+m^2)}{2} - 2\mu^2 \mathrm{dc}^{\,2}(\mu\,\xi, m)\,, \tag{102}$$

$$u = d_0 + d_1 y = c \pm 2\sqrt{a_4} y = c \pm 2\sqrt{\mu^2}\,\mathrm{dc}\,(\mu\,\xi, m)\,. \tag{103}$$

where A and μ are constants.

4 Coupled Nonlinear Klein–Gordon Schrödinger Equations

Coupled nonlinear Klein–Gordon Schrödinger equations reads

$$\begin{aligned} &u_{tt} - c_0^2 u_{xx} + f_0^2 u - \gamma |v|^2 = 0\,, \\ &\mathrm{i} v_t + \alpha v_{xx} + \beta u v = 0\,. \end{aligned} \tag{104}$$

We solve Eqs. (104) in the following frame,

$$u = u(\xi)\,, \quad v = \phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx - \omega t)}\,, \quad \xi = p(x - c_g t)\,. \tag{105}$$

Substituting Eq. (105) into Eqs. (104) leads to

$$\begin{aligned} &p^2(c_g^2 - c_0^2) u'' + f_0^2 u - \gamma \phi^2 = 0\,, \\ &\alpha p^2 \phi'' + ip(2\alpha k - c_g)\phi' + (\omega - \alpha k^2)\phi + \beta u \phi = 0\,. \end{aligned} \tag{106}$$

Set $c_g = 2\alpha k$, $\omega - \alpha k^2 = -\delta$, then one has

$$u'' + f_1 u - \gamma_1 \phi^2 = 0\,, \quad \phi'' - \delta_1 \phi + \beta_1 u \phi = 0\,, \tag{107}$$

where

$$\begin{aligned} &f_1 = \frac{f_0^2}{p^2(c_g^2 - c_0^2)}\,, \qquad \gamma_1 = \frac{\gamma}{p^2(c_g^2 - c_0^2)}\,, \\ &\delta_1 = \frac{\delta}{\alpha p^2}\,, \qquad \beta_1 = \frac{\beta}{\alpha p^2}\,. \end{aligned} \tag{108}$$

Similarly, we assume that the solutions of Eqs. (107) take the form of Eq. (65), we can get $n_1 = n_2 = 2$ for Eqs. (107), i.e.,

$$\begin{aligned} &u = b_0 + b_1 y + b_2 y^2\,, \quad \phi = d_0 + d_1 y + d_2 y^2\,, \\ &b_2 \neq 0\,, \qquad d_2 \neq 0\,, \end{aligned} \tag{109}$$

where y satisfies elliptic equation (3). Then substituting Eq. (109) into Eqs. (107) leads to

$$\begin{aligned} &b_2 = -\frac{6a_4}{\beta_1}\,, \quad b_1 = -\frac{3a_3}{\beta_1}\,, \\ &b_0 = \frac{1}{\beta_1}\Big[\delta_1 + \frac{f_1}{2} + \frac{3a_3^2}{8a_4} - 2a_2\Big]\,, \end{aligned} \tag{110}$$

and

$$\begin{aligned} &d_2 = \pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}\,, \quad d_1 = \pm\frac{3a_3}{\sqrt{-\beta_1\gamma_1}}\,, \\ &d_0 = \pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big[2a_2 + \frac{f_1}{2} - \frac{3a_3^2}{8a_4}\Big]. \end{aligned} \tag{111}$$

If $a_3 = 0$, then $b_1 = d_1 = a_1 = 0$ and

$$\begin{aligned} &b_0 = \frac{1}{\beta_1}\Big[\delta_1 + \frac{f_1}{2} - 2a_2\Big], \\ &d_0 = \pm\frac{1}{\sqrt{-\beta_1\gamma_1}}\Big[2a_2 + \frac{f_1}{2}\Big], \end{aligned} \tag{112}$$

then the transformation takes the following form,

$$y'^2 = a_0 + a_2 u^2 + a_4 u^4\,. \tag{113}$$

This is elliptic equation, and it also has many more kinds of solutions, some of which we will show next.

Case A Consider $a_0 = 0$, then we have two kinds of solutions.

(i) If $a_2 > 0$ and $a_4 > 0$, the solution is

$$y = \pm\sqrt{\frac{a_2}{a_4}}\,\mathrm{csch}\,(\sqrt{a_2}\,\xi)\,, \tag{114}$$

and

$$u = b_0 - \frac{6a_4}{\beta_1} y^2 = b_0 - \frac{6a_2}{\beta_1}\,\mathrm{csch}^2(\sqrt{a_2}\,\xi)\,, \tag{115}$$

$$\begin{aligned} v &= \Big[d_0 \pm \frac{6a_4}{\sqrt{-\beta_1\gamma_1}} y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)} \\ &= \Big[d_0 \pm \frac{6a_2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{csch}^2(\sqrt{a_2}\,\xi)\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)}\,. \end{aligned} \tag{116}$$

(ii) If $a_2 > 0$ and $a_4 < 0$, the solution is

$$y = \pm\sqrt{-\frac{a_2}{a_4}}\,\mathrm{sech}\,(\sqrt{a_2}\,\xi)\,, \tag{117}$$

and

$$u = b_0 - \frac{6a_4}{\beta_1} y^2 = b_0 + \frac{6a_2}{\beta_1}\,\mathrm{sech}^2(\sqrt{a_2}\,\xi)\,, \tag{118}$$

$$\begin{aligned} v &= \Big[d_0 \pm \frac{6a_4}{\sqrt{-\beta_1\gamma_1}} y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)} \\ &= \Big[d_0 \pm \frac{6a_2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{sech}^2(\sqrt{a_2}\,\xi)\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)}\,. \end{aligned} \tag{119}$$

Case B The ansatz just takes the form of Eq. (113), and there exist many more kinds of solutions expressed in terms of different Jacobi elliptic functions.[21] We show some generalized solutions just like what we have done in the former section next.

(i) If $a_0 = \mu^2 A^2$, $a_2 = -\mu^2(1 + m^2)$, and $a_4 = \mu^2 m^2 / A^2$, then the solution is

$$y = A\,\mathrm{sn}\,(\mu\,\xi, m)\,, \tag{120}$$

and

$$u = b_0 - \frac{6a_4}{\beta_1} y^2 = b_0 - \frac{6\mu^2 m^2}{\beta_1}\,\mathrm{sn}^2(\mu\,\xi, m)\,, \tag{121}$$

$$\begin{aligned} v &= \Big[d_0 \pm \frac{6a_4}{\sqrt{-\beta_1\gamma_1}} y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)} \\ &= \Big[d_0 \pm \frac{6\mu^2 m^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{sn}^2(\mu\,\xi, m)\Big]\,\mathrm{e}^{\mathrm{i}(kx - \omega t)}\,. \end{aligned} \tag{122}$$

(ii) If $a_0 = \mu^2(1 - m^2)A^2$, $a_2 = \mu^2(2m^2 - 1)$, and $a_4 = -\mu^2 m^2 / A^2$, then the solution is

$$y = A\,\mathrm{cn}\,(\mu\,\xi, m)\,, \tag{123}$$

and

$$u = b_0 - \frac{6a_4}{\beta_1} y^2 = b_0 + \frac{6\mu^2 m^2}{\beta_1}\,\mathrm{cn}^2(\mu\,\xi, m)\,, \tag{124}$$

$$v=\Big[d_0\pm\frac{6\mu^2m^2}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{cn}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{125}$$

(iii) If $a_0=\mu^2(1-m^2)A^2$, $a_2=\mu^2(2-m^2)$, and $a_4=-\mu^2/A^2$, then the solution is
$$y=A\,\mathrm{dn}\,(\mu\,\xi,m)\,, \tag{126}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0+\frac{6\mu^2}{\beta_1}\,\mathrm{dn}^2(\mu\,\xi,m)\,, \tag{127}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{dn}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{128}$$

(iv) If $a_0=\mu^2m^2A^2$, $a_2=-\mu^2(1+m^2)$, and $a_4=\mu^2/A^2$, then the solution is
$$y=A\,\mathrm{ns}\,(\mu\,\xi,m)\,, \tag{129}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2}{\beta_1}\,\mathrm{ns}^2(\mu\,\xi,m)\,, \tag{130}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{ns}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{131}$$

(v) If $a_0=-\mu^2m^2A^2$, $a_2=\mu^2(2m^2-1)$, and $a_4=\mu^2(1-m^2)/A^2$, then the solution is
$$y=A\,\mathrm{nc}\,(\mu\,\xi,m)\,, \tag{132}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2(1-m^2)}{\beta_1}\,\mathrm{nc}^2(\mu\,\xi,m)\,, \tag{133}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2(1-m^2)}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{nc}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{134}$$

(vi) If $a_0=-\mu^2A^2$, $a_2=\mu^2(2-m^2)$, and $a_4=\mu^2(m^2-1)/A^2$, then the solution is
$$y=A\,\mathrm{nd}\,(\mu\,\xi,m)\,, \tag{135}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2(m^2-1)}{\beta_1}\,\mathrm{nd}^2(\mu\,\xi,m)\,, \tag{136}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2(m^2-1)}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{nd}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{137}$$

(vii) If $a_0=\mu^2A^2$, $a_2=\mu^2(2-m^2)$, and $a_4=\mu^2(1-m^2)/A^2$, then the solution is
$$y=A\,\mathrm{sc}\,(\mu\,\xi,m)\,, \tag{138}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2(1-m^2)}{\beta_1}\,\mathrm{sc}^2(\mu\,\xi,m)\,, \tag{139}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2(1-m^2)}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{sc}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{140}$$

(viii) If $a_0=\mu^2A^2$, $a_2=\mu^2(2m^2-1)$, and $a_4=\mu^2(m^2-1)m^2/A^2$, then the solution is
$$y=A\,\mathrm{sd}\,(\mu\,\xi,m)\,, \tag{141}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2(m^2-1)m^2}{\beta_1}\,\mathrm{sd}^2(\mu\,\xi,m)\,, \tag{142}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2(m^2-1)m^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{sd}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{143}$$

(ix) If $a_0=\mu^2(1-m^2)A^2$, $a_2=\mu^2(2-m^2)$, and $a_4=\mu^2/A^2$, then the solution is
$$y=A\,\mathrm{cs}\,(\mu\,\xi,m)\,, \tag{144}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2}{\beta_1}\,\mathrm{cs}^2(\mu\,\xi,m)\,, \tag{145}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{cs}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{146}$$

(x) If $a_0=\mu^2A^2$, $a_2=-\mu^2(1+m^2)$, and $a_4=\mu^2m^2/A^2$, then the solution is
$$y=A\,\mathrm{cd}\,(\mu\,\xi,m)\,, \tag{147}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2m^2}{\beta_1}\,\mathrm{cd}^2(\mu\,\xi,m)\,, \tag{148}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=\Big[d_0\pm\frac{6\mu^2m^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{cd}^2(\mu\,\xi,m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \tag{149}$$

(xi) If $a_0=\mu^2m^2(m^2-1)A^2$, $a_2=\mu^2(2m^2-1)$, and $a_4=\mu^2/A^2$, then the solution is
$$y=A\,\mathrm{ds}\,(\mu\,\xi,m)\,, \tag{150}$$
and
$$u=b_0-\frac{6a_4}{\beta_1}y^2=b_0-\frac{6\mu^2}{\beta_1}\,\mathrm{ds}^2(\mu\,\xi,m)\,, \tag{151}$$
$$v=\Big[d_0\pm\frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[d_0 \pm \frac{6\mu^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{ds}^2(\mu\,\xi, m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \qquad (152)$$

(xii) If $a_0 = \mu^2 m^2 A^2$, $a_2 = -\mu^2(1+m^2)$, and $a_4 = \mu^2/A^2$, then the solution is

$$y = A\,\mathrm{dc}\,(\mu\,\xi, m)\,, \qquad (153)$$

and

$$u = b_0 - \frac{6a_4}{\beta_1}y^2 = b_0 - \frac{6\mu^2}{\beta_1}\,\mathrm{dc}^2(\mu\,\xi, m)\,, \qquad (154)$$

$$v = \Big[d_0 \pm \frac{6a_4}{\sqrt{-\beta_1\gamma_1}}y^2\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$= \Big[d_0 \pm \frac{6\mu^2}{\sqrt{-\beta_1\gamma_1}}\,\mathrm{dc}^2(\mu\,\xi, m)\Big]\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,. \qquad (155)$$

Of course, we can have more solutions if we do not take $a_3 = 0$, but we do not discuss this here for sententiousness. In this section, we got more kinds of periodic solutions, solitary wave solutions, rational solutions, and so on. Moreover, we got more kinds of envelope periodic solutions, envelope solitary wave solutions, envelope rational solutions, and so on. Similarly, these solutions can be applied to explain some nonlinear phenomena, especially in optical fibre communications and others.

5 Conclusion

In this paper, we consider elliptic equation as a new transformation and propose a new method to solve coupled nonlinear systems, more kinds of solutions can be got there, including rational solutions, solitary wave solutions constructed in terms of hyperbolic functions, periodic solutions expressed in terms of trigonometric functions, and periodic solutions dealing with elliptic functions. Here we got more new periodic solutions expressed in terms of function of Jacobi elliptic sine function and/or Jacobi elliptic cosine function and/or Jacobi elliptic function of the third kind, these solutions are not given in literatures to our knowledge.

If $a_4 = 1$ and $a_0 = a_2^2/4$ in Eq. (69) or (113), then

$$y' = \frac{a_2}{2} + y^2\,, \qquad (156)$$

which just recovers transformation (1) given by Fan.[8] And if we take $a_0 = R^2$, $a_2 = 2\mu R^2$, and $a_4 = \mu^2 R^2$, then transformation (69) or (113) also recovers the transformation (2) given by Yan.[10] So it is obvious that transformations (1) and (2) are just special cases of Eq. (3). But, applying transformations (1) and (2) to solve nonlinear wave equations, the periodic solutions expressed in terms of elliptic functions cannot be obtained. And application of transformation (3) to some coupled nonlinear systems, the obtained solutions consist of those from the hyperbolic tangent expansion method,[8,9,11] the Jacobi elliptic function expansion method,[12,13] the nonlinear transformation method,[14,15] and the trial function method,[16,17] so it can be taken as a unified method, and more applications to solve other nonlinear wave equations are also applicable.

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