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Lame function and multi-order exact solutions to nonlinear evolution equations

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Abstract

In this paper, based on the Lame function and Jacobi elliptic function, the perturbation method is applied to some nonlinear evolution equations, and there many multi-order solutions are derived to these nonlinear evolution equations.

1. Introduction

To find the exact solutions of nonlinear evolution equations plays an important role in nonlinear studies. Applying some new methods, such as the homogeneous balance method [1–3], the hyperbolic tangent function expansion method [4–6], the nonlinear transformation method [7,8], the trial function method [9,10], sine–cosine method [11], the Jacobi elliptic function expansion method [12,13] and so on [14–16], many exact solutions are obtained, from which rich structures are shown to exist in different nonlinear wave equations. Furthermore, in order to discuss the stability of theses solutions, one must superimpose a small disturbance on these solutions and analyze the evolution of the small disturbance [18,19]. This is equivalent to the solutions of nonlinear evolution equations expanded as a power series in terms of a small parameter ϵ and then multi-order exact solutions are derived. In this paper, using the Jacobi elliptic function expansion method, the multi-order exact solutions of some nonlinear evolution equations are obtained by means of the Jacobi elliptic functions and Lame function [17,18].

2. Lame functions

Usually, Lame equation [17] in terms of $y(x)$ can be written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+[\lambda-n(n+1)m^2\,\mathrm{sn}^2x]y=0 \tag{1}$$

where λ is an eigenvalue, n is a positive integer, $\mathrm{sn}\,x$ is the Jacobi elliptic sine function with its modulus m $(0<m<1)$.
Set

$$\eta=\mathrm{sn}^2x \tag{2}$$

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then the Lame equation (1) becomes

$$\frac{d^2y}{d\eta^2}+\frac{1}{2}\left(\frac{1}{\eta}+\frac{1}{\eta-1}+\frac{1}{\eta-h}\right)\frac{dy}{d\eta}-\frac{\mu+n(n+1)\eta}{4\eta(\eta-1)(\eta-h)}y=0 \tag{3}$$

where

$$h=m^{-2}>1,\quad \mu=-h\lambda \tag{4}$$

Eq. (3) is a kind of Fuchs-typed equations with four regular singular points $\eta=0,1,h$ and $\eta=\infty$, the solution to Lame equation (3) is known as Lame function.

For example, when $n=3$, $\lambda=4(1+m^2)$, i.e. $\mu=-4(1+m^{-2})$, the Lame function is

$$L_3(x)=\eta^{1/2}(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{sn}x\,\mathrm{cn}x\,\mathrm{dn}x \tag{5}$$

When $n=2$, $\lambda=1+m^2$, i.e. $\mu=-(1+m^{-2})$, the Lame function is

$$L_2(x)=(1-\eta)^{1/2}(1-h^{-1}\eta)^{1/2}=\mathrm{cn}x\,\mathrm{dn}x \tag{6}$$

In Eqs. (5) and (6), $\mathrm{cn}x$ and $\mathrm{dn}x$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind [17,18], respectively. In the next sections, we will apply these two kinds of Lame functions $L_3(x)$ and $L_2(x)$ to solve nonlinear evolution equations and to derive their corresponding multi-order exact solutions.

3. Multi-order exact solutions with $L_3(x)$

In this case, the Lame equation (1) reduces to

$$\frac{d^2y}{dx^2}+[4(1+m^2)-12m^2\,\mathrm{sn}^2x]y=0 \tag{7}$$

here $n=3$ and $\lambda=4(1+m^2)$ is chosen for (1) and the solution to (7) is (5). Next, we will illustrate the application of (7) to solve some nonlinear evolution equations.

3.1. Boussinesq equation

Boussinesq equation reads

$$\frac{\partial^2u}{\partial t^2}-c_0^2\frac{\partial^2u}{\partial x^2}-\alpha\frac{\partial^4u}{\partial x^4}-\beta\frac{\partial^2u^2}{\partial x^2}=0 \tag{8}$$

We seek its travelling wave solutions of the following form

$$u=u(\xi),\quad \xi=k(x-ct) \tag{9}$$

where k and c are wave number and wave speed, respectively.

Substituting (9) into (8), we have

$$\alpha k^2\frac{d^4u}{d\xi^4}+\beta\frac{d^2u^2}{d\xi^2}-(c^2-c_0^2)\frac{d^2u}{d\xi^2}=0 \tag{10}$$

Integrating (10) twice with respect to ξ and taking the integration constants as zero, we get

$$\alpha k^2\frac{d^2u}{d\xi^2}+\beta u^2-(c^2-c_0^2)u=0 \tag{11}$$

Here we consider perturbation method and setting

$$u=u_0+\epsilon u_1+\epsilon^2u_2+\cdots \tag{12}$$

where $\epsilon(0<\epsilon\ll 1)$ is a small parameter, u_0, u_1 and u_2 represent the zeroth-order, first-order and second-order solutions, respectively.

Substituting (12) into (11), we derive the following systems of the zeroth-order, the first-order and the second-order equations:

$$\epsilon^0: \quad \alpha k^2 \frac{d^2 u_0}{d\xi^2} + \beta u_0^2 - (c^2 - c_0^2) u_0 = 0 \tag{13}$$

$$\epsilon^1: \quad \alpha k^2 \frac{d^2 u_1}{d\xi^2} + [2\beta u_0 - (c^2 - c_0^2)] u_1 = 0 \tag{14}$$

and

$$\epsilon^2: \quad \alpha k^2 \frac{d^2 u_2}{d\xi^2} + [2\beta u_0 - (c^2 - c_0^2)] u_2 = -\beta u_1^2 \tag{15}$$

The zeroth-order equation (13) can be solved by the Jacobi elliptic sine function expansion method, the ansatz solution

$$u_0 = a_0 + a_1 \operatorname{sn}\xi + a_2 \operatorname{sn}^2\xi \tag{16}$$

can be assumed.

Substituting (16) into (13), the expansion coefficients a_0, a_1 and a_2 can be easily determined as

$$a_0 = \frac{c^2 - c_0^2}{2\beta} + \frac{6\alpha}{\beta}(1 + m^2)k^2, \quad a_1 = 0, \quad a_2 = -\frac{6\alpha}{\beta} m^2 k^2 \tag{17}$$

so the zeroth-order exact solution is

$$u_0 = \frac{c^2 - c_0^2}{2\beta} + \frac{6\alpha}{\beta}(1 + m^2)k^2 - \frac{6\alpha}{\beta} m^2 k^2 \operatorname{sn}^2\xi \tag{18}$$

Substituting the zeroth-order exact solution (18) into the first-order equation (14) yields

$$\frac{d^2 u_1}{d\xi^2} + [4(1 + m^2) - 12 m^2 \operatorname{sn}^2\xi] u_1 = 0 \tag{19}$$

obviously this is just a Lame equation as (7) with $n = 3$ and $\lambda = 4(1 + m^2)$, so its solution is

$$u_1 = A L_3(\xi) = A \operatorname{sn}\xi \operatorname{cn}\xi \operatorname{dn}\xi \tag{20}$$

where A is an arbitrary constant and (20) is the first-order exact solution of Boussinesq equation (8).

In order to solve the second-order equation (15), the zeroth-order exact solution (18) and the first-order exact solution (20) have to be substituted into (15), thus the second-order equation (15) is rewritten as

$$\frac{d^2 u_2}{d\xi^2} + [4(1 + m^2) - 12 m^2 \operatorname{sn}^2\xi] u_2 = -\frac{\beta A^2}{\alpha k^2} \operatorname{sn}^2\xi \operatorname{cn}^2\xi \operatorname{dn}^2\xi \tag{21}$$

it is obvious that this is an inhomogeneous Lame equation with $n = 3$ and $\lambda = 4(1 + m^2)$. Its solution of homogeneous equation is just the same one as (20) and its special solution of inhomogeneous terms can be assumed to be

$$u_2 = b_0 + b_2 \operatorname{sn}^2\xi + b_4 \operatorname{sn}^4\xi \tag{22}$$

Substituting (22) into (21), we can determine the expansion coefficients b_0, b_2 and b_4 as

$$b_0 = -\frac{\beta A^2}{24 m^2 \alpha k^2}, \quad b_2 = \frac{(1 + m^2)\beta A^2}{12 m^2 \alpha k^2}, \quad b_4 = -\frac{\beta A^2}{8 \alpha k^2} \tag{23}$$

so the second-order exact solution of Boussinesq equation (8) can be written as

$$u_2 = -\frac{\beta A^2}{24 m^2 \alpha k^2} [1 - 2(1 + m^2) \operatorname{sn}^2\xi + 3 m^2 \operatorname{sn}^4\xi] \tag{24}$$

3.2. KdV equation

KdV equation reads

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{25}$$

Substituting (9) into (25) yields

$$\beta k^2 \frac{d^3 u}{d\xi^3} + u\frac{du}{d\xi} - c\frac{du}{d\xi} = 0 \tag{26}$$

Integrating (26) once with respect to ξ and taking integration constant as zero, we have

$$\beta k^2 \frac{d^2 u}{d\xi^2} + \frac{1}{2}u^2 - cu = 0 \tag{27}$$

Substituting (12) into (27), we get the zeroth-order, the first-order and the second-order equations:

$$\epsilon^0: \quad \beta k^2 \frac{d^2 u_0}{d\xi^2} + \frac{1}{2}u_0^2 - cu_0 = 0 \tag{28}$$

$$\epsilon^1: \quad \beta k^2 \frac{d^2 u_1}{d\xi^2} + (u_0 - c)u_1 = 0 \tag{29}$$

and

$$\epsilon^2: \quad \beta k^2 \frac{d^2 u_2}{d\xi^2} + (u_0 - c)u_2 = -\frac{1}{2}u_1^2 \tag{30}$$

Applying (16) to (18), the zeroth-order exact solution can be easily obtained

$$u_0 = c + 4(1+m^2)\beta k^2 - 12m^2\beta k^2 \operatorname{sn}^2\xi \tag{31}$$

Similarly, substituting (31) into the first-order equation (29) leads to

$$\frac{d^2 u_1}{d\xi^2} + [4(1+m^2) - 12m^2 \operatorname{sn}^2\xi]u_1 = 0 \tag{32}$$

obviously this is the Lame equation, its solution is

$$u_1 = A \operatorname{sn}\xi \operatorname{cn}\xi \operatorname{dn}\xi \tag{33}$$

where A is an arbitrary constant.

Substituting the zeroth-order solution (31) and the first-order solution (33) into the second-order equation (30) results in

$$\frac{d^2 u_2}{d\xi^2} + [4(1+m^2) - 12m^2 \operatorname{sn}^2\xi]u_2 = -\frac{A^2}{2\beta k^2}\operatorname{sn}^2\xi \operatorname{cn}^2\xi \operatorname{dn}^2\xi \tag{34}$$

Then applying (22) to (34), the second-order exact solution of KdV equation (25) can be written as

$$u_2 = -\frac{A^2}{48m^2\beta k^2}[1 - 2(1+m^2)\operatorname{sn}^2\xi + 3m^2\operatorname{sn}^4\xi] \tag{35}$$

4. Multi-order exact solutions with $L_2(x)$

In this case, the Lame equation (1) reduces to

$$\frac{d^2 y}{dx^2} + [(1+m^2) - 6m^2 \operatorname{sn}^2 x]y = 0 \tag{36}$$

here $n = 2$ and $\lambda = 1 + m^2$ is chosen for (1) and the solution to (36) is (6). Next, we will illustrate the application of (36) to solve some other nonlinear evolution equations.

4.1. mBBM equation

Modified Benjamin–Bona–Mahony (mBBM) equation reads

$$\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{37}$$

Seeking its travelling wave solution in the frame of (9), so we have

$$(c_0 - c)\frac{du}{d\xi} + u^2\frac{du}{d\xi} - \beta k^2 c\frac{d^3u}{d\xi^3} = 0 \tag{38}$$

which can be integrated once with respect to ξ and the integration constant is taken to be zero to reach its another form

$$\beta k^2 c\frac{d^2u}{d\xi^2} - \frac{1}{3}u^3 + (c - c_0)u = 0 \tag{39}$$

Considering the perturbation method and (12) and (39) can be expanded as multi-order equations and the first three order equations are

$$\epsilon^0: \quad \beta k^2 c\frac{d^2u_0}{d\xi^2} - \frac{1}{3}u_0^3 + (c - c_0)u_0 = 0 \tag{40}$$

$$\epsilon^1: \quad \beta k^2 c\frac{d^2u_1}{d\xi^2} - [u_0^2 - (c - c_0)]u_1 = 0 \tag{41}$$

and

$$\epsilon^2: \quad \beta k^2 c\frac{d^2u_2}{d\xi^2} - [u_0^2 - (c - c_0)]u_2 = u_0u_1^2 \tag{42}$$

From the zeroth-order equation (40) and the ansatz solution

$$u_0 = a_0 + a_1 \operatorname{sn}\xi \tag{43}$$

we can get the zeroth-order exact solution of mBBM equation

$$u_0 = \sqrt{6m^2\beta c}k \operatorname{sn}\xi, \quad c - c_0 = (1 + m^2)\beta k^2 c \tag{44}$$

Substituting the zeroth-order exact solution (44) into the first-order equation (41) leads to

$$\frac{d^2u_1}{d\xi^2} + [(1 + m^2) - 6m^2 \operatorname{sn}^2\xi]u_1 = 0 \tag{45}$$

which takes the same form as Lame equation (36), so the first-order exact solution can be written as

$$u_1 = AL_2(\xi) = A \operatorname{cn}\xi \operatorname{dn}\xi \tag{46}$$

where A is an arbitrary constant.

Substituting the zeroth-order exact solution (44) and the first-order exact solution (46) into the second-order equation (41) results in

$$\frac{d^2u_2}{d\xi^2} + [(1 + m^2) - 6m^2 \operatorname{sn}^2\xi]u_2 = \pm\sqrt{\frac{6}{\beta c}}\frac{mA^2}{k}\operatorname{sn}\xi \operatorname{cn}^2\xi \operatorname{dn}^2\xi \tag{47}$$

which is an inhomogeneous Lame equation of the form (36), and it can be solved by introducing an ansatz solution

$$u_2 = b_1 \operatorname{sn}\xi + b_3 \operatorname{sn}^3\xi \tag{48}$$

Combining (47) with (48) reaches the second-order exact solution

$$u_2 = \mp\sqrt{\frac{6}{\beta c}}\frac{(1 + m^2)A^2}{12mk}\operatorname{sn}\xi\left[1 - \frac{2m^2}{1 + m^2}\operatorname{sn}^2\xi\right] \tag{49}$$

4.2. mKdV equation

mKdV equation reads

$$\frac{\partial u}{\partial t} + \alpha u^2\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{50}$$

In the frame of (9) and (50) can be written as

$$\beta k^2 \frac{d^2 u}{d\xi^2} + \frac{\alpha}{3}u^3 - cu = 0 \tag{51}$$

where integration with respect to ξ has been taken once and the integration constant is set as zero.

Applying the perturbation method to (51), we can derive the zeroth-order, the first-order and the second-order equations as

$$\epsilon^0: \quad \beta k^2 \frac{d^2 u_0}{d\xi^2} + \frac{\alpha}{3}u_0^3 - cu_0 = 0 \tag{52}$$

$$\epsilon^1: \quad \beta k^2 \frac{d^2 u_1}{d\xi^2} + (\alpha u_0^2 - c)u_1 = 0 \tag{53}$$

and

$$\epsilon^2: \quad \beta k^2 \frac{d^2 u_2}{d\xi^2} + (\alpha u_0^2 - c)u_2 = -\alpha u_0 u_1^2 \tag{54}$$

Similarly, from (43) and the zeroth-order equation (52), the zeroth-order exact solution is derived as

$$u_0 = \pm\sqrt{-\frac{6\beta}{\alpha}} mk\, \mathrm{sn}\,\xi \tag{55}$$

Substituting (55) into the first-order equation (53) leads to the first-order exact solution

$$u_1 = A\, \mathrm{cn}\,\xi\, \mathrm{dn}\,\xi \tag{56}$$

where A is an arbitrary constant.

Combining (48), (55) and (56) with (54) gives the second-order exact solution of mKdV

$$u_2 = \mp\sqrt{-\frac{6\alpha}{\beta}} \frac{(1+m^2)A^2}{12mk} \mathrm{sn}\,\xi \left[1 - \frac{2m^2}{1+m^2}\mathrm{sn}^2\,\xi\right] \tag{57}$$

5. Conclusion and discussion

In this paper, the Lame equation and Lame functions are applied to solve nonlinear evolution equations. When perturbation method and two kinds of Lame functions $L_3(x)$ and $L_2(x)$ are considered, then the multi-order solutions are obtained for these nonlinear evolution equations. The results obtained in this paper are very important for nonlinear instability analysis of nonlinear coherent structures.

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