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Basic Pattern in Atmospheric Turbulence Model*

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Abstract *From the controlling equations of atmosphere motion, Prandtl's mixing length theory is used to derive the atmospheric turbulence models, such as Burgers equation model and Burgers-KdV equation model. And then the projective Riccati equations are applied to solve these atmospheric turbulence models, where much more patterns are obtained, including solitary wave pattern, singular pattern, and so on.*

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1 Introduction

If the fluctuation of density is neglected, for an inviscid and adiabatic atmosphere the averaged equations of motion and thermodynamic equation can be written as[1]

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}t} - f\bar{v} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial x} - \Big(\frac{\partial\overline{u'^2}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\Big), \tag{1a}$$

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + f\bar{u} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial y} - \Big(\frac{\partial\overline{v'u'}}{\partial x} + \frac{\partial\overline{v'^2}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}\Big), \tag{1b}$$

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}t} = g - \frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial z} - \Big(\frac{\partial\overline{w'u'}}{\partial x} + \frac{\partial\overline{w'v'}}{\partial y} + \frac{\partial\overline{w'^2}}{\partial z}\Big), \tag{1c}$$

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t} = -\Big(\frac{\partial\overline{\theta'u'}}{\partial x} + \frac{\partial\overline{\theta'v'}}{\partial y} + \frac{\partial\overline{\theta'w'}}{\partial z}\Big) \tag{1d}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z}, \tag{2}$$

where $\bar{()}$ and $()'$ denote the average quantities and fluctuating quantities, respectively, g is acceleration due to gravity, and f is Coriolis parameter.

If extended Prandtl's mixing length theory is applied, then the fluctuating quantities can be expressed in terms of corresponding averaged quantities, for example,

$$\begin{aligned} u' &= \bar{u}(z_0) - \bar{u}(z) \approx -\frac{\partial\bar{u}}{\partial z}(z-z_0) - \frac{1}{2}\frac{\partial^2\bar{u}}{\partial z^2}(z-z_0)^2 \\ &= -l'\frac{\partial\bar{u}}{\partial z} - \frac{1}{2}l'^2\frac{\partial^2\bar{u}}{\partial z^2}, \end{aligned} \tag{3}$$

where $l' = z - z_0$ is a mixing length. Thus

$$-\overline{u'w'} = \nu\frac{\partial\bar{u}}{\partial z} - \beta\frac{\partial^2\bar{u}}{\partial z^2} \tag{4}$$

with $\nu = \overline{l'w'}$ and $\beta = -\overline{l'^2w'}/2$, which are called eddy viscosity coefficient and eddy dispersion coefficient, respectively. So from Eq. (4), one has

$$-\frac{\partial\overline{u'w'}}{\partial z} = \nu\frac{\partial^2\bar{u}}{\partial z^2} - \beta\frac{\partial^3\bar{u}}{\partial z^3}. \tag{5}$$

If the assumption of homogeneous motion in various directions is taken, then from Eqs. (1) we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u - \beta\nabla\cdot\Box u, \tag{6a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v - \beta\nabla\cdot\Box v, \tag{6b}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - \beta\nabla\cdot\Box w, \tag{6c}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu\nabla^2\theta - \kappa\nabla\cdot\Box\theta \tag{6d}$$

with

$$\begin{aligned} \nabla &= \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}, \\ \Box &= \boldsymbol{i}\frac{\partial^2}{\partial x^2} + \boldsymbol{j}\frac{\partial^2}{\partial y^2} + \boldsymbol{k}\frac{\partial^2}{\partial z^2}, \end{aligned} \tag{7}$$

where the averaged symbol is omitted, μ and κ are coefficients of thermal diffusivity and thermal dispersion, respectively.

If the hydrostatic equilibrium satisfies in vertical direction in the lowest-order and Coriolis force and the pressure gradient force in horizontal directions can be neglected, then for the higher-order motions, equations (6) reduce approximatively to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \nu\nabla^2 u - \beta\nabla\cdot\Box u, \tag{8a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \nu\nabla^2 v - \beta\nabla\cdot\Box v, \tag{8b}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \nu\nabla^2 w - \beta\nabla\cdot\Box w, \tag{8c}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu\nabla^2\theta - \kappa\nabla\cdot\Box\theta. \tag{8d}$$

We can see that the above four equations take the same form, and that if only x-direction is considered, equation (8a) reduces to KdV-Burgers equation,[2]

$$u_t + uu_x - \nu u_{xx} + \beta u_{xxx} = 0. \tag{9}$$

If classical Prandtl's mixing length theory is applied, i.e. $\beta = 0$, then we have Burgers equation,[3,4]

$$u_t + uu_x - \nu u_{xx} = 0. \tag{10}$$

Equation (10) was firstly proposed by Burgers[4] to study issues of turbulence, and the interaction between nonlinearity and dissipation can result in different coherent structures such as vortex sheets, pancakes, and filaments.[3] Of course, the interaction between nonlinearity and dissipation can also lead to turbulence and chaos,

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and it is generally recognized that turbulence in its fully developed stage has a singular structure. But, in a fluid system with dispersion low dissipation (large Reynolds number) cannot necessarily lead to irregular turbulent motion, there maybe exist some coherent structures. So equation (9) was applied by Liu[5−7] to model the inverse energy cascade[8] and intermittent turbulence, where dispersion effect is taken into account.

A number of problems are described in terms of suitable nonlinear models in branches of physics, mathematics, and other interdisciplinary sciences, such as nonlinear Schrödinger equations in plasma physics,[9] KdV equation in shallow water model,[2] and so on. It is an interesting topic to seek exact solutions to these nonlinear models. Contrary to the integrable nonlinear equations, non-integrable nonlinear equations are more difficult to be solved. Especially, if odd-order derivatives and even-order derivatives exist simultaneously in a nonlinear equation, then it is much more difficult to derive solutions to them. So new methods are needed for this kind of equations, for many methods do not work yet.

In the next sections, we will introduce a transformation and then apply this transformation to solve the above Burgers and Burgers-KdV equations. Then we will derive many travelling wave solutions to these Burgers-type equations, among which are solitary wave solutions.

2 Solutions to Burgers Equation

In order to solve the above Burgers-type equations, the travelling wave frame is chosen, i.e.

$$\xi = x - ct\,. \tag{11}$$

Substituting Eq. (11) into Burgers equation (10) yields

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - \nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = 0\,, \tag{12}$$

which can be integrated once and rewritten as

$$-cu + \frac{u^2}{2} - \nu\frac{\mathrm{d}u}{\mathrm{d}\xi} = A\,, \tag{13}$$

where A is an integration constant.

In order to solve Eq. (13) we introduce the following crucial ansatz

$$u(\xi) = \sum_{i=1}^{n} f^{i-1}(\xi)[a_i f(\xi) + b_i g(\xi)] + a_0\,, \quad a_n^2 + b_n^2 \neq 0\,, \tag{14}$$

where n can be determined by balancing the highest order derivative term with the high degree nonlinear term in Eq. (13). And f and g are solutions to the well-known projective Riccati equations,[10−12]

$$f'(\xi) = pf(\xi)g(\xi)\,, \tag{15a}$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi)\,, \tag{15b}$$

where $p \neq 0$ is a real constant, q and r are two real constants. When $p = -1$ and $q = 1$, equations (15) reduce to the coupled equations given in Refs. [10] and [11], and when $p = \pm 1$ and $q \geq 0$, equations (15) reduce to the coupled equations given in Ref. [12]. There is a relation between f and g

$$g^2 = -\frac{1}{p}\Big[q - 2rf + \frac{r^2+\delta}{q}f^2\Big]\,, \tag{16}$$

where $\delta = \pm 1$.

Apply the above expansion method, if we take the expansion order of u as $O(u) = n$ and consider the relations (15), then $O(\mathrm{d}u/\mathrm{d}\xi) = n+1$, so partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 1$. Obviously, the formal solution can be written as

$$u = a_0 + a_1 f(\xi) + b_1 g(\xi), \quad a_1^2 + b_1^2 \neq 0\,. \tag{17}$$

Considering the relation (16), from Eq. (17) one can have

$$u' = b_1 rf - \frac{b_1(r^2+\delta)}{q}f^2 + a_1 pfg\,, \tag{18}$$

$$u^2 = \Big(a_0^2 - \frac{b_1^2 q}{p}\Big) + 2\Big(a_0 a_1 + \frac{b_1^2 r}{p}\Big)f + 2a_0 b_1 g + \Big[a_1^2 - \frac{b_1^2(r^2+\delta)}{pq}\Big]f^2 + 2a_1 b_1 fg\,. \tag{19}$$

Substituting Eqs. (17) $\sim$ (19) into Eq. (13) results in the following algebraic equations,

$$-\frac{1}{2}a_0^2 + \frac{q}{2p}b_1^2 + ca_0 = A\,, \tag{20a}$$

$$\nu rb_1 - \Big(a_0 a_1 + \frac{r}{p}b_1^2\Big) + ca_1 = 0\,, \tag{20b}$$

$$-a_0 b_1 + cb_1 = 0\,, \tag{20c}$$

$$-\frac{\nu(r^2+\delta)}{q}b_1 - \frac{1}{2}a_1^2 + \frac{r^2+\delta}{2pq}b_1^2 = 0\,, \tag{20d}$$

$$\nu pa_1 - a_1 b_1 = 0\,. \tag{20e}$$

From Eqs. (20), four kinds of solutions can be derived. The first one is

$$a_0 = c, \quad a_1 = 0, \quad b_1 = 2\nu p, \quad r = 0, \tag{21}$$

the second one is

$$a_0 = c, \quad a_1 = 0, \quad b_1 = \nu p, \quad r = \pm 1 \quad (\delta = -1)\,, \tag{22}$$

the third one is

$$a_0 = c, \quad a_1 = \pm\sqrt{-\frac{\nu^2 p\delta}{q}}, \quad b_1 = \nu p, \quad r = 0\,, \tag{23}$$

and the last one is

$$a_0 = c, \quad a_1 = \pm\sqrt{-\frac{\nu^2 p(r^2+\delta)}{q}}, \quad b_1 = \nu p\,. \tag{24}$$

For the projective Riccati equations (15), when $pq < 0$ and $\delta = 1$, their solutions are

$$f_1 = \frac{q}{r + \sinh(\sqrt{-pq}\,\xi)}\,, \tag{25}$$

$$g_1 = -\frac{\sqrt{-pq}}{p}\frac{\cosh(\sqrt{-pq}\,\xi)}{r + \sinh(\sqrt{-pq}\,\xi)}\,, \tag{26}$$

and when $pq < 0$ and $\delta = -1$, their solutions are

$$f_2 = \frac{q}{r + \cosh(\sqrt{-pq}\,\xi)}\,, \tag{27}$$

$$g_2 = -\frac{\sqrt{-pq}}{p}\frac{\sinh(\sqrt{-pq}\,\xi)}{r + \cosh(\sqrt{-pq}\,\xi)}\,. \tag{28}$$

When $pq > 0$ and $\delta = -1$, their solutions are

$$f_3 = \frac{q}{r + \sin(\sqrt{pq}\,\xi)}\,, \tag{29}$$

$$g_3 = -\frac{\sqrt{pq}}{p}\frac{\cos(\sqrt{pq}\,\xi)}{r+\sin(\sqrt{pq}\,\xi)}\,, \tag{30}$$

and

$$f_4 = \frac{q}{r+\cos(\sqrt{pq}\,\xi)}\,, \tag{31}$$

$$g_4 = \frac{\sqrt{pq}}{p}\frac{\sin(\sqrt{pq}\,\xi)}{r+\cos(\sqrt{pq}\,\xi)}\,. \tag{32}$$

Combining Eq. (13), Eq. (17), and results from Eqs. (21) $\sim$ (32), we can derive various travelling solutions including solitary wave solutions to Burgers equation (10). For example, for the first kind of solutions (21), the following three cases must be considered.

Case 1 $pq < 0$ and $\delta = 1$, the solution is

$$u_1 = c - 2\nu\sqrt{-pq}\,\coth\sqrt{-pq}\,\xi\,. \tag{33}$$

Case 2 $pq < 0$ and $\delta = -1$, the solution is

$$u_2 = c - 2\nu\sqrt{-pq}\,\tanh\sqrt{-pq}\,\xi\,. \tag{34}$$

Case 3 $pq > 0$ and $\delta = -1$, the solutions are

$$u_3 = c - 2\nu\sqrt{pq}\,\cot\sqrt{pq}\,\xi \tag{35}$$

and

$$u_4 = c + 2\nu\sqrt{pq}\,\tan\sqrt{pq}\,\xi\,. \tag{36}$$

The above four solutions have been obtained by other methods, such as Ref. [13].

For the second kind of solutions (22), the following two cases must be considered.

Case 1 $pq < 0$ and $\delta = -1$, the solution is

$$u_{5,6} = c - \nu\frac{\sqrt{-pq}\,\sinh\sqrt{-pq}\,\xi}{\cosh\sqrt{-pq}\,\xi \pm 1}\,. \tag{37}$$

Case2 $pq > 0$ and $\delta = -1$, the solutions are

$$u_{7,8} = c - \nu\frac{\sqrt{pq}\,\cos\sqrt{pq}\,\xi}{\sin\sqrt{pq}\,\xi \pm 1} \tag{38}$$

and

$$u_{9,10} = c + \nu\frac{\sqrt{pq}\,\sin\sqrt{pq}\,\xi}{\cos\sqrt{pq}\,\xi \pm 1}\,. \tag{39}$$

The above six solutions have not been obtained by other methods, they are new solutions.

For the third kind of solutions (23), the following two cases must be considered.

Case 1 $pq < 0$ and $\delta = 1$, the solution is

$$u_{11,12} = c - \nu\sqrt{-pq}\frac{\coth\sqrt{-pq}\,\xi \pm 1}{\sinh\sqrt{-pq}\,\xi}\,. \tag{40}$$

Case2 $pq > 0$ and $\delta = -1$, the solutions are

$$u_{13,14} = c - \nu\sqrt{pq}\,\frac{\cos\sqrt{pq}\,\xi \pm 1}{\sin\sqrt{pq}\,\xi} \tag{41}$$

and

$$u_{15,16} = c + \nu\sqrt{pq}\,\frac{\sin\sqrt{pq}\,\xi \pm 1}{\cos\sqrt{pq}\,\xi}\,. \tag{42}$$

The above six solutions have not been obtained by other methods, either.

For the last kind of solutions (24), the following two cases must be considered.

Case 1 $pq < 0$ and $\delta = 1$, the solution is

$$u_{17,18} = c \pm \sqrt{-\frac{\nu^2 p(r^2+1)}{q}}\frac{q}{\sinh\sqrt{-pq}\,\xi + r} - \nu\sqrt{-pq}\frac{\cosh\sqrt{-pq}\,\xi}{\sinh\sqrt{-pq}\,\xi + r}\,. \tag{43}$$

Case 2 $pq < 0$, $\delta = -1$ and $r^2 > 1$, the solution is

$$u_{19,20} = c \pm \sqrt{-\frac{\nu^2 p(r^2-1)}{q}}\frac{q}{\cosh\sqrt{-pq}\,\xi + r} - \nu\sqrt{-pq}\frac{\sinh\sqrt{-pq}\,\xi}{\cosh\sqrt{-pq}\,\xi + r}\,. \tag{44}$$

Case 3 $pq > 0$, $\delta = -1$ and $r^2 < 1$, the solutions are

$$u_{21,22} = c \pm \sqrt{\frac{\nu^2 p(1-r^2)}{q}}\frac{q}{\sin\sqrt{pq}\,\xi + r} - \nu\sqrt{pq}\,\frac{\cos\sqrt{pq}\,\xi}{\sin\sqrt{pq}\,\xi + r} \tag{45}$$

and

$$u_{23,24} = c \pm \sqrt{\frac{\nu^2 p(1-r^2)}{q}}\frac{q}{\cos\sqrt{pq}\,\xi + r} + \nu\sqrt{pq}\,\frac{\sin\sqrt{pq}\,\xi}{\cos\sqrt{pq}\,\xi + r}\,. \tag{46}$$

The above eight solutions have not been obtained by other methods, either.

3 Solutions to Burgers-KdV Equation

In the frame of Eq. (11), Burgers-KdV equation (9) reduces to

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - \nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} + \beta\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = 0\,. \tag{47}$$

Integrating Eq. (47) once yields

$$-cu + \frac{1}{2}u^2 - \nu\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = A\,, \tag{48}$$

where A is an integration constant.

Similarly, combining Eq. (48) with series expansion (14) yields $n = 2$, i.e., the formal solution to Eq. (48) is

$$u = a_0 + a_1 f + b_1 g + a_2 f^2 + b_2 fg, \quad a_2^2 + b_2^2 \neq 0\,. \tag{49}$$

So

$$u' = (b_1 r - b_2 q)f + \Big[3b_2 r - \frac{b_1(r^2+\delta)}{q}\Big]f^2 + a_1 pfg - \frac{2b_2(r^2+\delta)}{q}f^3 + 2a_2 pf^2 g\,, \tag{50}$$

$$\begin{aligned} u^2 = {} & \Big(a_0^2 - \frac{b_1^2 q}{p}\Big) + \Big(2a_0a_1 + \frac{2b_1^2 r}{p} - \frac{2b_1b_2q}{p}\Big)f + 2a_0b_1 g \\ & + \Big[a_1^2 - \frac{b_1^2(r^2+\delta)}{pq} + 2a_0a_2 + \frac{4b_1b_2r}{p} - \frac{b_2^2 q}{p}\Big]f^2 + [2a_1b_1 + 2a_0b_2]fg \\ & + \Big[2a_1a_2 - \frac{2b_1b_2(r^2+\delta)}{pq} + \frac{2b_2^2 r}{p}\Big]f^3 + (2b_1a_2 + 2a_1b_2)f^2 g \\ & + \Big[a_2^2 - \frac{b_2^2(r^2+\delta)}{pq}\Big]f^4 + 2a_2b_2f^3g\,, \end{aligned} \tag{51}$$

$$u'' = -a_1pqf + (3a_1pr - 4a_2pq)f^2 + (b_1r - b_2q)pfg$$

$$+\left[10a_2pr-\frac{2a_1p(r^2+\delta)}{q}\right]f^3$$
$$+\left[6b_2pr-\frac{2b_1p(r^2+\delta)}{q}\right]f^2g$$
$$-\frac{6a_2p(r^2+\delta)}{q}f^4-\frac{6b_2p(r^2+\delta)}{q}f^3g\,. \tag{52}$$

Substituting Eqs. (49) and (52) into Eq. (48) results in the following algebraic equations

$$-ca_0+\frac{1}{2}\Big(a_0^2-\frac{q}{p}b_1^2\Big)=A\,, \tag{53a}$$
$$-ca_1+\Big(a_0a_1+\frac{r}{p}b_1^2-\frac{q}{p}b_1b_2\Big)-\nu rb_1$$
$$+\nu qb_2-\beta pqa_1=0\,, \tag{53b}$$
$$-cb_1+a_0b_1=0\,, \tag{53c}$$
$$-ca_2+\frac{1}{2}\Big[a_1^2-\frac{(r^2+\delta)}{pq}b_1^2+2a_0a_2+\frac{4r}{p}b_1b_2-\frac{q}{p}b_2^2\Big]$$
$$-3\nu rb_2+\frac{\nu(r^2+\delta)}{q}b_1+3\beta pra_1$$
$$-4\beta pqa_2=0\,, \tag{53d}$$
$$-cb_2+a_1b_1+a_0b_2-\nu pa_1+\beta prb_1$$
$$-\beta pqb_2=0\,, \tag{53e}$$
$$a_1a_2-\frac{(r^2+\delta)}{pq}b_1b_2+\frac{r}{p}b_2^2+\frac{2\nu(r^2+\delta)}{q}b_2$$
$$+10\beta pra_2-\frac{2\beta p(r^2+\delta)}{q}a_1=0\,, \tag{53f}$$
$$b_1a_2+a_1b_2-2\nu pa_2+6\beta prb_2$$
$$-\frac{2\beta p(r^2+\delta)}{q}b_1=0\,, \tag{53g}$$
$$\frac{1}{2}a_2^2-\frac{(r^2+\delta)}{2pq}b_2^2-\frac{6\beta p(r^2+\delta)}{q}a_2=0\,, \tag{53h}$$
$$a_2b_2-\frac{6\beta p(r^2+\delta)}{q}a_2=0\,. \tag{53i}$$

Solving Eqs. (53) yields two kinds of solutions. The first one is

$$a_0=c,\quad a_1=0,\quad a_2=12\beta p\delta/q\,,$$
$$b_1=12p\nu/5,\quad b_2=0,\quad r=0 \tag{54}$$

with the constraint

$$pq=-\nu^2/100\beta^2<0\,. \tag{55}$$

The second one is

$$a_0=c,\quad a_1=\pm 6p\beta,\quad a_2=6\beta p/q,\quad b_1=6p\nu/5\,,$$
$$b_2=\pm 6p\nu/5q,\quad r=0,\quad \delta=1 \tag{56}$$

with the constraint

$$pq=-\nu^2/25\beta^2<0\,. \tag{57}$$

For the first kind of solutions (55), two cases must be considered.

Case 1 $\delta=1$, the solution is

$$u_1=c+\frac{6\nu^2}{25\beta}-\frac{3\nu^2}{25\beta}\Big(\coth\frac{\nu}{10\beta}\xi+1\Big)^2\,. \tag{58}$$

Case 2 $\delta=-1$, the solution is

$$u_2=c+\frac{6\nu^2}{25\beta}-\frac{3\nu^2}{25\beta}\Big(\tanh\frac{\nu}{10\beta}\xi+1\Big)^2\,. \tag{59}$$

For the second kind of solutions (56), only one case needs be considered. The solution is

$$u_3=c-\frac{6\nu^2}{25\beta}\Big[\coth\frac{\nu}{5\beta}\xi\pm\mathrm{csch}\frac{\nu}{5\beta}\xi+\mathrm{csch}^2\frac{\nu}{5\beta}\xi$$
$$\pm\coth\frac{\nu}{5\beta}\xi\,\mathrm{csch}\frac{\nu}{5\beta}\xi\Big]\,. \tag{60}$$

In the above solutions, u_1 and u_2 have been found by other method, such as Ref. [13], but u_3 has not been found yet.

4 Conclusion

In this paper, we first apply Prandtl's mixing length theory to derive Burgers equation and Burgers-KdV equation from the controlling equations of atmosphere motion. And then we introduce an intermediate transformation to solve these Burgers-type nonlinear equations. Based on the results from simplified nonlinear equations, solutions to the Burgers-type nonlinear equations are obtained, where general travelling wave solutions and solitary wave solutions are given. Compared with results obtained from other methods,[14–16] more new solutions are derived in this paper. These new solutions correspond to different basic patterns that can be formed in atmospheric motions.

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