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Exact Periodic-Wave Solutions for (2+1)-Dimensional Boussinesq Equation and (3+1)-Dimensional KP Equation*

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Abstract *The (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation are studied by using the extended Jacobi elliptic-function method. The exact periodic-wave solutions for the two equations are obtained.*

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1 Introduction

The construction of exact solutions of nonlinear evolution equations in mathematical physics plays an important role in understanding the nonlinear problems. In recent years, important progress has been made in understanding the higher-dimensional nonlinear partial differential equations (PDEs), especially in (2+1)- and (3+1)-dimensions.[1,2] Particularly, various powerful methods have been used to explore different kinds of solutions of various physical models described by nonlinear PDEs, such as the inverse scattering transform method,[1] Hirota method,[3] tanh method,[4] sine-cosine method,[5] homogeneous balance method (HBM),[6–8] and Lie group analysis,[9–11] etc. Recently, the Jacobi elliptic-function methods for finding periodic-wave solutions to nonlinear evolution equations were proposed in Refs. [12] $\sim$ [14]. Senthilvelan[15] and Chen *et al.*[16] studied the travelling wave solutions for the (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation by HBM and explored certain solutions of the equations. In this letter, we would like to further discuss the (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation by our extended Jacobi elliptic-function method. As a result, exact periodic-wave solutions and solitary wave solutions to the two equations are obtained.

2 Method

Consider a given PDE of the form

$$N(u, u_t, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yz}, u_{xt}, \ldots) = 0\,, \tag{1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, etc. In this letter, we seek the following formal travelling wave solutions to Eq. (1),

$$u(x, y, z, t) = u(\xi), \quad \xi = kx + ly + sz - \omega t\,, \tag{2}$$

where $(k,\ l,\ s)$ are the components of the wave-number vector in the x, y, and z directions respectively, and ω is the angular frequency. Substituting Eq. (2) into Eq. (1) yields an ordinary differential equation (ODE) for $u(\xi)$ with constant coefficients,

$$N(u, u', u'', u''', \ldots) = 0\,, \tag{3}$$

where prime denotes differentiation with respect to ξ. Equation (3) can be integrated as long as all terms contain derivatives (this is true for the equations considered here). In this process we take the integration constants to be zero. The next crucial step is to express the solutions of the resulting ODE by the Jacobi elliptic-function method in Ref. [12], $u(\xi)$ can be expressed as a finite power series of Jacobi elliptic sine function, $\operatorname{sn}\xi$, i.e., the ansatz

$$u(\xi) = \sum_{j=0}^{n} a_j \operatorname{sn}^j \xi, \quad a_n \neq 0\,. \tag{4}$$

We assume the degree of $u(\xi)$ as $O(u(\xi)) = n$, which leads to the degrees of other expressions in Eq. (3) as

$$O\Big(\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\Big) = n + p, \quad O\Big(u^q \frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\Big) = (q+1)n + p\,,$$
$$q = 0, 1, 2, \ldots, \quad p = 1, 2, 3, \ldots \tag{5}$$

Notice that

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \sum_{j=0}^{n} j a_j \operatorname{sn}^{j-1}\xi \operatorname{cn}\xi \operatorname{dn}\xi\,, \tag{6}$$

where $\operatorname{cn}\xi$ and $\operatorname{dn}\xi$ are Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind, respectively. And

$$\operatorname{cn}^2\xi = 1 - \operatorname{sn}^2\xi, \quad \operatorname{dn}^2\xi = 1 - m^2 \operatorname{sn}^2\xi\,, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{sn}\xi = \operatorname{cn}\xi \operatorname{dn}\xi\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{cn}\xi = -\operatorname{sn}\xi \operatorname{dn}\xi\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\operatorname{dn}\xi = -m^2 \operatorname{sn}\xi \operatorname{cn}\xi\,. \tag{8}$$

In this article, for Jacobi elliptic functions, we use the notation $\operatorname{sn}\xi$, $\operatorname{cn}\xi$, $\operatorname{dn}\xi$ with argument ξ and modulus parameter m $(0 < m < 1)$. The parameter n in Eq. (4) will be fixed by balancing the highest order of derivative term and the nonlinear term in the nonlinear ODE Eq. (3) by using Eq. (5). Substituting Eq. (4) (with fixed value of n) into the reduced nonlinear ODE (3) and equating the coefficients of various powers of $\operatorname{sn}\xi$ to zero we get a set of algebraic equations for a_j, k, l, s, and ω. Solving them consistently we obtain relations among the parameters a_j,

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k, l, s, and ω. If any parameters are left unspecified, they are regarded as being arbitrary constants. Making use of these relations we can find a final expression for $u(\xi)$ which leads to an expression for the travelling wave solutions for Eq. (1).

We also wish to mention that instead of the $\mathrm{sn}\,\xi$ in the ansatz (4) one can choose the cn- or dn-function method in Refs. [13] and [14]. We only mention it here without going into details.

3 Solutions

3.1 *(2+1)-Dimensional Boussinesq Equation*

Let us consider a (2+1)-dimensional Boussinesq equation,[17]

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} - u_{yy} = 0\,, \tag{9}$$

which is introduced to describe the propagation of gravity waves on the surface of water, in particular the head-on collision of oblique waves. This equation combines the two-way propagation of the classical Boussinesq equation with the dependence on a second spatial variable, as occurs in the two-dimensional Korteweg-de Vries (KdV) equation, also known as Kadomstev–Petviashvili (KP) equation. Equation (9) is not a completely integrable equation, although it still provides a description of head-on collision of oblique waves and it does possess some interesting properties.[17] In Ref. [17], exact and general solitary-wave, two-soliton, and resonant solutions of Eq. (9) are obtained by Hirota's bilinear method. In this letter, we will find exact periodic-wave solutions to Eq. (9). As described in Sec. 2, we seek for a travelling wave solution $u(x,y,t) = u(\xi)$, $\xi = kx + ly - \omega t$. Substituting $u(\xi)$ into Eq. (9) yields

$$\begin{aligned} &k^4\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4} + (k^2+l^2-\omega^2)\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} - 6k^2u\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} \\ &\qquad - 6k^2\Big(\frac{\mathrm{d}u}{\mathrm{d}\xi}\Big)^2 = 0\,, \end{aligned} \tag{10}$$

$$O\Big(u\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}\Big) = 2n+2, \quad O\Big(\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4}\Big) = n+4\,. \tag{11}$$

Considering Eq. (11) to balance the highest order derivative with the nonlinear terms in Eq. (10), we get $n = 2$, and the solution of Eq. (9) in terms of $\mathrm{sn}\,\xi$ is

$$u(\xi) = a_0 + a_1\,\mathrm{sn}\,\xi + a_2\,\mathrm{sn}^2\xi\,. \tag{12}$$

Integrating Eq. (10) with respect to ξ twice yields

$$k^4\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} - 3k^2u^2 + (k^2+l^2-\omega^2)u = 0\,. \tag{13}$$

Substituting Eq. (12) into Eq. (13) yields

$$\begin{aligned} &3k^2(2m^2k^2 - a_2)a_2\mathrm{sn}^4\xi + 2k^2(m^2k^2 - 3a_2)a_1\mathrm{sn}^3\xi \\ &\quad + \{[(k^2+l^2-\omega^2) - 4k^4(1+m^2)]a_2 \\ &\quad - 3k^2(a_1^2 + 2a_0a_2)\}\mathrm{sn}^2\xi \\ &\quad + [(k^2+l^2-\omega^2) - k^4(1+m^2) - 6k^2a_0]a_1\,\mathrm{sn}\,\xi \\ &\quad + 2k^4a_2 - 3k^2a_0^2 + (k^2+l^2-\omega^2)a_0 = 0\,. \end{aligned} \tag{14}$$

Setting each coefficient of $\mathrm{sn}^n(\xi)$ $(n = 0,1,2,3,4)$ to zero yields a set of equations for a_0, a_1, a_2, k, l, and ω. From the solution of these equations under condition $a_2 \neq 0$, $k \neq 0$, $l \neq 0$, and $\omega \neq 0$, the coefficients are determined as

$$\begin{aligned} &a_0 = \frac{k^2+l^2-\omega^2}{6k^2} - \frac{2}{3}(1+m^2)k^2\,, \\ &a_1 = 0, \quad a_2 = 2m^2k^2\,. \end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (12), a final solution is given as

$$\begin{aligned} u(x,y,t) &= \frac{k^2+l^2-\omega^2}{6k^2} - \frac{2}{3}(1+m^2)k^2 + 2m^2k^2\mathrm{sn}^2\xi \\ &= \frac{k^2+l^2-\omega^2}{6k^2} - \frac{2}{3}(1-2m^2)k^2 \\ &\quad - 2m^2k^2\mathrm{cn}^2(kx+ly-\omega t)\,, \end{aligned} \tag{16}$$

which is the exact periodic-wave solution to Eq. (9). Usually, it is known as the cnoidal wave solution of the (2+1)-dimensional Boussinesq equation. The behavior of $u(x,y,t)$ in Eq. (16) for the parameters $m = 0.95$, $k = l = \omega = 1$ is illustrated in Fig. 1.

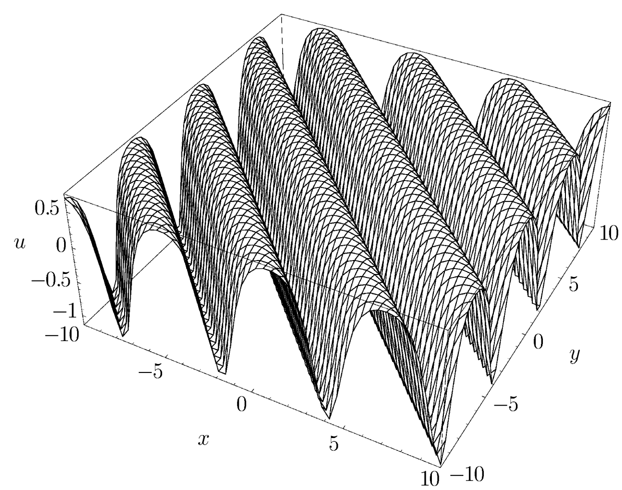

Fig. 1 $u(x,y,t=0)$ vs. (x,y) for $m = 0.95$, $k = l = \omega = 1$.

For $m \to 1$, $\mathrm{cn}\,\xi \to \mathrm{sech}\,\xi$, thus equation (16) degenerated as the following form,

$$\begin{aligned} u(x,y,t) &= \frac{k^2+l^2-\omega^2}{6k^2} + \frac{2}{3}k^2 \\ &\quad - 2k^2\mathrm{sech}^2(kx+ly-\omega t)\,. \end{aligned} \tag{17}$$

This is the solitary wave solution of Eq. (9).

3.2 *(3+1)-Dimensional KP Equation*

Let us now consider the (3+1)-dimensional KP equation,

$$u_{xt} - 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0\,. \tag{18}$$

We firstly make the formal travelling wave transformation Eq. (2). Substituting $u(\xi)$ into Eq. (18) and integrating it with respect to ξ twice yields

$$k^4\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} - 3k^2u^2 + (k\omega + l^2 + s^2)u = 0\,. \tag{19}$$

According to the same steps as the above-mentioned ones, its corresponding ansatz solution is Eq. (12). Similarly, exact periodic-wave solutions to Eq. (18) can be obtained,

$$\begin{aligned} u(x,y,z,t) &= \frac{k\omega + l^2 + s^2}{6k^2} - \frac{2}{3}(1-2m^2)k^2 - 2m^2k^2\mathrm{cn}^2\xi \\ &= \frac{k\omega + l^2 + s^2}{6k^2} - \frac{2}{3}(1-2m^2)k^2 \\ &\quad - 2m^2k^2\mathrm{cn}^2(kx+ly+sz-\omega t)\,, \end{aligned} \tag{20}$$

and its corresponding solitary wave solution is

$$\begin{aligned} u(x,y,z,t) = \frac{k\omega + l^2 + s^2}{6k^2} + \frac{2}{3}k^2 \\ - 2k^2\,\mathrm{sech}^2(kx+ly+sz-\omega t)\,. \end{aligned} \tag{21}$$

4 Conclusions

Exact periodic-wave solutions and solitary wave solutions for both (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation have been obtained by our extended Jacobi elliptic-function method. This method is very simple and powerful. It has three obvious advantages: (i) only some algebra is needed to obtain exact periodic-wave solutions to the equations under consideration; (ii) the Jacobi elliptic-functions can be easily manipulated by the symbolic computation software, *Mathematica* or *Maple*, which allows us to perform complicated deducing and tedious algebraic calculation on a computer and output directly the required solutions; (iii) the periodic-wave solutions for the two equations can also be obtained by making appropriate linear superpositions of known periodic solutions. This unusual procedure for generating solutions is successful as a consequence of some powerful, recently discovered, cyclic identities by the Jacobi elliptic functions. The interested reader is referred to Refs. [18] $\sim$ [20] for more details. The extended Jacobi elliptic-function method as described in Sec. 2 can be also applied to other higher-dimensional nonlinear PDEs, for example, the cylindrical KP equation

$$\begin{aligned} &(u_t + 6uu_x + u_{xxx})_x + \frac{1}{2t}u_x + \frac{3\sigma^2}{t^2}u_{yy} = 0\,, \\ &\sigma^2 = \pm 1\,, \end{aligned} \tag{22}$$

which is also known as the Johnson's equation and describes cylindrical solitons in an ideal, inviscid fluid.[21] Another example is Zakharov–Kuznetsov (ZK) equation,

$$u_t + uu_x + (\partial_x^2 + \partial_y^2)u_x = 0\,, \tag{23}$$

which was first derived by Zakharov and Kuznetsov to describe nonlinear ion-acoustic waves in a strongly magnetized plasma.[22] The ZK equation can be thought of as a generalization of the KdV equation to two spatial dimensions but, unlike the Kadomstev–Petviashvili equation, it is not integrable by the inverse scattering transform method.[23] The (2+1)-dimensional NLS equation[24] is

$$\mathrm{i}\psi_t + \psi_{xx} + \sigma_d\psi_{yy} + 2\sigma_n|\psi|^2\psi = 0\,, \tag{24}$$

where $\sigma_n = \pm 1$ defines the type of the cubic nonlinearity, i.e., focusing (at $\sigma_n = +1$) or defocusing (at $\sigma_n = -1$), and $\sigma_d = \pm 1$ defines the type of the wave dispersion/diffraction. We only mention it here without going into details.

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