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Multiple structures of two-dimensional nonlinear Rossby wave

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Abstract

In this paper, the elliptic equation is taken as a transformation and applied to solve the Zakharov–Kuznetsov equation, which has been derived by Gottwald as a two-dimensional model for nonlinear Rossby waves. It is shown that more kinds of solutions are derived, such as periodic solutions of rational form, periodic solutions and so on.
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1. Introduction

Gottwald derived the nonlinear dispersive Zakharov–Kuznetsov equation from the quasigeostrophic barotropic vorticity equation [1], where weakly nonlinear and long wave multiple scale analysis were applied to two dimensions and derived the Zakharov–Kuznetsov (ZK for short) equation for nonlinear Rossby waves. In contrast to the Kadometsev–Petviashvilli (KP for short) equation, the ZK equation is first derived in a geophysical fluid dynamics context. At the same time, the ZK equation supports stable lump solitary waves [2], which makes the ZK equation a very attractive model equation for the study of vortices in geophysical flows. Actually, there are more multiple structures in the ZK equations, we will show next. All these studies may help to describe two-dimensional coherent structures such as atmospheric blocking events, long lived eddies in the ocean or coherent structures in the Jovian atmosphere such as the Great Red Spot.

We have taken elliptic equation as an intermediate transformation to solve nonlinear wave equations [3–5], and obtained many periodic solutions and solitary wave solutions. However, there are still more research needed to do in order to find more solutions of different forms. In the reference [6], we derived periodic solutions of rational forms, which are due to external forcing. In this paper, the elliptic equation is taken as a transformation and applied to solve the ZK equation to multiple structures of two-dimensional nonlinear Rossby waves.

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2. Multiple structures of two-dimensional nonlinear Rossby waves

ZK equation [1] reads

$$u_t + \delta u_x + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = 0 \tag{1}$$

which was derived by Gottwald from the quasigeostrophic vorticity equation as a two-dimensional model for nonlinear Rossby waves.

We seek its travelling wave solutions in the following frame

$$u = u(\xi), \quad \xi = kx + ly - \omega t \tag{2}$$

here ω is angular frequency, k and l are wave number in x and y direction, respectively.

Substituting Eq. (2) into Eq. (1) and integrating once yield

$$-(\omega - \delta k)u + \frac{\alpha k}{2}u^2 + (\beta k^3 + \gamma k l^2)u'' = C \tag{3}$$

where C is an integration constant. And then we suppose Eq. (3) has the following solution

$$u = u(z) = \sum_{j=0}^{j=n} b_j z^j, \quad z = z(\xi), \quad b_n \neq 0 \tag{4}$$

where z satisfies the elliptic equation [7]

$$z'^2 = \sum_{i=0}^{i=4} a_i z^i, \quad a_4 \neq 0 \tag{5}$$

where $z' = \frac{dz}{d\xi}$, then

$$z'' = \frac{a_1}{2} + a_2 z + \frac{3a_3}{2}z^2 + 2a_4 z^3 \tag{6}$$

Obviously, two special cases of (5) are

$$\frac{dz}{d\xi} = b + z^2 \tag{7}$$

and

$$\frac{dz}{d\xi} = R(1 + \mu z^2) \tag{8}$$

which were introduced by Fan [8] and Yan et al. [9], respectively.

There n in the Eq. (4) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in the Eq. (3). Here we know that the degree of u is

$$\mathrm{O}(u) = \mathrm{O}(z^n) = n \tag{9}$$

and from (5) and (6), one has

$$\mathrm{O}(z'^2) = \mathrm{O}(z^4) = 4, \quad \mathrm{O}(z'') = \mathrm{O}(z^3) = 3 \tag{10}$$

and actually one can have

$$\mathrm{O}(z^{(d)}) = d + 1 \tag{11}$$

So one has

$$\mathrm{O}(u) = n, \quad \mathrm{O}(u') = n + 1, \quad \mathrm{O}(u'') = n + 2, \quad \mathrm{O}(u^{(d)}) = n + d \tag{12}$$

For ZK equation (1), we have $n = 2$, so the ansatz solution of (4) can be rewritten as

$$u = b_0 + b_1 z + b_2 z^2, \quad b_2 \neq 0 \tag{13}$$

then

$$u^2 = b_0^2 + 2b_0 b_1 z + (2b_0 b_2 + b_1^2)z^2 + 2b_1 b_2 z^3 + b_2^2 z^4 \tag{14}$$

$$u'' = \left(\frac{1}{2}a_1b_1 + 2a_0b_2\right) + (a_2b_1 + 3a_1b_2)z + \left(\frac{3}{2}a_3b_1 + 4a_2b_2\right)z^2 + (2a_4b_1 + 5a_3b_2)z^3 + 6a_4b_2z^4 \tag{15}$$

Substituting Eqs. ((13)–(15)) into Eq. (3) and collecting each order of z yield algebraic equations about coefficients $b_j(j = 0, 1, 2)$ and $a_i(i = 0, 1, 2, 3, 4)$, i.e.

$$-(\omega - \delta k)b_0 + \frac{\alpha k}{2}b_0^2 + (\beta k^3 + \gamma k l^2)\left(\frac{1}{2}a_1b_1 + 2a_0b_2\right) - C = 0 \tag{16a}$$

$$-(\omega - \delta k)b_1 + \alpha k b_0 b_1 + (\beta k^3 + \gamma k l^2)(a_2b_1 + 3a_1b_2) = 0 \tag{16b}$$

$$-(\omega - \delta k)b_2 + \frac{\alpha k}{2}(2b_0b_2 + b_1^2) + (\beta k^3 + \gamma k l^2)\left(\frac{3}{2}a_3b_1 + 4a_2b_2\right) = 0 \tag{16c}$$

$$\alpha k b_1 b_2 + (\beta k^3 + \gamma k l^2)(2a_4b_1 + 5a_3b_2) = 0 \tag{16d}$$

$$\frac{\alpha k}{2}b_2^2 + 6(\beta k^3 + \gamma k l^2)a_4b_2 = 0 \tag{16e}$$

from which we have

$$\begin{aligned} b_2 &= -\frac{12(\beta k^2 + \gamma l^2)}{\alpha}a_4, \quad b_1 = -\frac{(6\beta k^2 + \gamma l^2)}{\alpha}a_3, \\ b_0 &= \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)}{\alpha}a_2 + \frac{3(\beta k^2 + \gamma l^2)a_3^2}{4\alpha a_4} \end{aligned} \tag{17}$$

at the same time there is

$$a_1 = \frac{a_3}{2a_4}\left(a_2 - \frac{a_3^2}{4a_4}\right) \tag{18}$$

So if $a_3 = 0$, then

$$b_1 = a_1 = 0, \quad b_2 = -\frac{12(\beta k^2 + \gamma l^2)}{\alpha}a_4, \quad b_0 = \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)}{\alpha}a_2 \tag{19}$$

and the transformation (5) takes the following form

$$z'^2 = a_0 + a_2z^2 + a_4z^4 \tag{20}$$

which has many kinds of solutions, next we will show some solutions of rational forms expressed in terms of different elliptic functions [7].

(1) If $a_0 = (1-m^2)/4$, $a_2 = (1 + m^2)/2$ and $a_4 = (1-m^2)/4$(where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions, see [7,10–12]), then the solutions to Eq. (20) are

$$z_1 = \frac{\mathrm{cn}(\xi, m)}{1 + \mathrm{sn}(\xi, m)} \tag{21}$$

where $\mathrm{sn}(\xi, m)$ and $\mathrm{cn}(\xi, m)$ are Jacobi elliptic sine and cosine function, respectively,(see [7,10–12]) and

$$z_2 = \frac{\mathrm{cn}(\xi, m)}{1 - \mathrm{sn}(\xi, m)} \tag{22}$$

These are two new solutions to Eq. (20) which are not shown in references [3–5]. So based on the above results, we can derive new solutions to Eq. (1)

$$u_1 = b_0 + b_2z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 + m^2) - \frac{3(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{cn}^2(\xi, m)}{\alpha[1 + \mathrm{sn}(\xi, m)]^2} \tag{23}$$

and

$$u_2 = b_0 + b_2z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 + m^2) - \frac{3(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{cn}^2(\xi, m)}{\alpha[1 - \mathrm{sn}(\xi, m)]^2} \tag{24}$$

(2) If $a_0 = -(1-m^2)/4$, $a_2 = (1+m^2)/2$ and $a_4 = -(1-m^2)/4$, then the solutions to Eq. (20) are

$$z_3 = \frac{\mathrm{dn}(\xi, m)}{1 + m\mathrm{sn}(\xi, m)} \tag{25}$$

and

$$z_4 = \frac{\mathrm{dn}(\xi, m)}{1 - m\mathrm{sn}(\xi, m)} \tag{26}$$

where $\mathrm{dn}(\xi,m)$ is Jacobi elliptic function of the third kind (see [7,10–12]) and new solutions to Eq. (1) are

$$u_3 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 + m^2) + \frac{3(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{dn}^2(\xi, m)}{\alpha[1 + m\mathrm{sn}(\xi, m)]^2} \tag{27}$$

and

$$u_4 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 + m^2) + \frac{3(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{dn}^2(\xi, m)}{\alpha[1 - m\mathrm{sn}(\xi, m)]^2} \tag{28}$$

(3) If $a_0 = m^2/4$, $a_2 = -(2-m^2)/2$ and $a_4 = m^2/4$, then the solutions to Eq. (20) are

$$z_5 = \frac{m\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)} \tag{29}$$

and

$$z_6 = \frac{m\mathrm{sn}(\xi, m)}{1 - \mathrm{dn}(\xi, m)} \tag{30}$$

and new solutions to Eq. (1) are

$$u_5 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(2 - m^2) - \frac{3(\beta k^2 + \gamma l^2)m^4\mathrm{sn}^2(\xi, m)}{\alpha[1 + \mathrm{dn}(\xi, m)]^2} \tag{31}$$

and

$$u_6 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(2 - m^2) - \frac{3(\beta k^2 + \gamma l^2)m^4\mathrm{sn}^2(\xi, m)}{\alpha[1 - \mathrm{dn}(\xi, m)]^2} \tag{32}$$

It is known that when $m \to 1$, $\mathrm{sn}(\xi,m) \to \tanh\xi$, $\mathrm{cn}(\xi,m) \to \mathrm{sech}\xi$, $\mathrm{dn}(\xi,m) \to \mathrm{sech}\xi$, so new solutions to Eq. (1) are

$$u_{5'} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha} - \frac{3(\beta k^2 + \gamma l^2)\tanh^2(\xi)}{\alpha[1 + \mathrm{sech}(\xi)]^2} \tag{33}$$

and

$$u_{6'} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha} - \frac{3(\beta k^2 + \gamma l^2)\tanh^2(\xi)}{\alpha[1 - \mathrm{sech}(\xi)]^2} \tag{34}$$

(4) If $a_0 = 1/4$, $a_2 = (1-2m^2)/2$ and $a_4 = 1/4$, then the solutions to Eq. (20) are

$$z_7 = \frac{\mathrm{sn}(\xi, m)}{1 + \mathrm{cn}(\xi, m)} \tag{35}$$

and

$$z_8 = \frac{\mathrm{sn}(\xi, m)}{1 - \mathrm{cn}(\xi, m)} \tag{36}$$

and new solutions to Eq. (1) are

$$u_7 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 - 2m^2) - \frac{3(\beta k^2 + \gamma l^2)\mathrm{sn}^2(\xi, m)}{\alpha[1 + \mathrm{cn}(\xi, m)]^2} \tag{37}$$

and

$$u_8 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(1 - 2m^2) - \frac{3(\beta k^2 + \gamma l^2)\mathrm{sn}^2(\xi, m)}{\alpha[1 - \mathrm{cn}(\xi, m)]^2} \tag{38}$$

Similarly, when $m \to 1$, the solutions u_7 and u_8 reduce to solutions $u_{5'}$ and $u_{6'}$.

(5) If $a_0 = 1/4$, $a_2 = -(2-m^2)/2$ and $a_4 = m^4/4$, then the solutions to Eq. (20) are

$$z_9 = \frac{\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)} \tag{39}$$

and

$$z_{10} = \frac{\mathrm{sn}(\xi, m)}{1 - \mathrm{dn}(\xi, m)} \tag{40}$$

and new solutions to Eq. (1) are

$$u_9 = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(2 - m^2) - \frac{3(\beta k^2 + \gamma l^2) m^4 \mathrm{sn}^2(\xi, m)}{\alpha [1 + \mathrm{dn}(\xi, m)]^2} \tag{41}$$

and

$$u_{10} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{2(\beta k^2 + \gamma l^2)}{\alpha}(2 - m^2) - \frac{3(\beta k^2 + \gamma l^2) m^4 \mathrm{sn}^2(\xi, m)}{\alpha [1 - \mathrm{dn}(\xi, m)]^2} \tag{42}$$

which are the same as (31) and (32) respectively.

Remark 1. The above ten solutions to Eq. (20) i.e. z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9 and z_{10}, are first applied to present the exact solutions to the nonlinear wave equations. The above twelve solutions to Eq. (1) are first reported here to our knowledge.

Apart from the above rational form solutions expressed in terms of different elliptic functions [7], there are many more kinds of solutions expressed in terms of different elliptic functions [7] directly, for there are more other kinds of solutions to Eq. (20). For example,

(11) If $a_0 = 1$, $a_2 = -(1 + m^2)$ and $a_4 = m^2$(where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions, see [10,7,11,12]), then the solution is

$$z_{11} = \mathrm{sn}(\xi, m) \tag{43}$$

and

$$u_{11} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 + m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2) m^2 \mathrm{sn}^2(\xi, m)}{\alpha} \tag{44}$$

(12) If $a_0 = m^2-1$, $a_2 = 2m^2-1$ and $a_4 = -m^2$, then the solution is

$$z_{12} = \mathrm{cn}(\xi, m) \tag{45}$$

and

$$u_{12} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 - 2m^2)}{\alpha} + \frac{12(\beta k^2 + \gamma l^2) m^2 \mathrm{cn}^2(\xi, m)}{\alpha} \tag{46}$$

(13) If $a_0 = 1-m^2$, $a_2 = 2-m^2$ and $a_4 = -1$, then the solution is

$$z_{13} = \mathrm{dn}(\xi, m) \tag{47}$$

and

$$u_{13} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)(2 - m^2)}{\alpha} + \frac{12(\beta k^2 + \gamma l^2) \mathrm{dn}^2(\xi, m)}{\alpha} \tag{48}$$

(14) If $a_0 = m^2$, $a_2 = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$z_{14} = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)} \tag{49}$$

and

$$u_{14} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 + m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2) \mathrm{ns}^2(\xi, m)}{\alpha} \tag{50}$$

(15) If $a_0 = -m^2$, $a_2 = 2m^2-1$ and $a_4 = 1-m^2$, then the solution is

$$z_{15} = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)} \tag{51}$$

and

$$u_{15} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 - 2m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{nc}^2(\xi, m)}{\alpha} \tag{52}$$

(16) If $a_0 = -1$, $a_2 = 2-m^2$ and $a_4 = m^2-1$, then the solution is

$$z_{16} = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)} \tag{53}$$

and

$$u_{16} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)(2 - m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)(m^2 - 1)\mathrm{nd}^2(\xi, m)}{\alpha} \tag{54}$$

(17) If $a_0 = 1$, $a_2 = 2-m^2$ and $a_4 = 1-m^2$, then the solution is

$$z_{17} = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)} \tag{55}$$

and

$$u_{17} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)(2 - m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)(1 - m^2)\mathrm{sc}^2(\xi, m)}{\alpha} \tag{56}$$

(18) If $a_0 = 1$, $a_2 = 2m^2-1$ and $a_4 = (m^2-1)m^2$, then the solution is

$$z_{18} = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)} \tag{57}$$

and

$$u_{18} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 - 2m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)(m^2 - 1)m^2\mathrm{sd}^2(\xi, m)}{\alpha} \tag{58}$$

(19) If $a_0 = 1-m^2$, $a_2 = 2-m^2$ and $a_4 = 1$, then the solution is

$$z_{19} = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)} \tag{59}$$

and

$$u_{19} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} - \frac{4(\beta k^2 + \gamma l^2)(2 - m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)\mathrm{cs}^2(\xi, m)}{\alpha} \tag{60}$$

(20) If $a_0 = 1$, $a_2 = -(1 + m^2)$ and $a_4 = m^2$, then the solution is

$$z_{20} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)} \tag{61}$$

and

$$u_{20} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 + m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)m^2\mathrm{cd}^2(\xi, m)}{\alpha} \tag{62}$$

(21) If $a_0 = m^2(m^2-1)$, $a_2 = 2m^2-1$ and $a_4 = 1$, then the solution is

$$z_{21} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)} \tag{63}$$

and

$$u_{21} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 - 2m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)\mathrm{ds}^2(\xi, m)}{\alpha} \tag{64}$$

(22) If $a_0 = m^2$, $a_2 = -(1 + m^2)$ and $a_4 = 1$, then the solution is

$$z_{22} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)} \tag{65}$$

and

$$u_{22} = b_0 + b_2 z^2 = \frac{\omega - \delta k}{\alpha k} + \frac{4(\beta k^2 + \gamma l^2)(1 + m^2)}{\alpha} - \frac{12(\beta k^2 + \gamma l^2)\mathrm{dc}^2(\xi, m)}{\alpha} \tag{66}$$

It is known that when $m \to 1$, $\mathrm{sn}(\xi,m) \to \tanh\xi$, $\mathrm{cn}(\xi,m) \to \mathrm{sech}\xi$, $\mathrm{dn}(\xi,m) \to \mathrm{sech}\xi$ and when $m \to 0$, $\mathrm{sn}(\xi,m) \to \sin\xi$, $\mathrm{cn}(\xi,m) \to \cos\xi$. And among the Jacobi elliptic functions, Jacobi elliptic sine function, Jacobi elliptic cosine function and Jacobi elliptic function of the third kind are three basic ones, all other Jacobi elliptic functions can be expressed in terms of them. So we also can get more solutions expressed in terms of hyperbolic functions and trigonometric functions.

Remark 2. Some of the solutions given above are singular. Actually, These solutions are a kind of solutions dealing with "hot spots" or "blow-up" of solutions [13–16], which can develop singularity at a finite point.

3. Conclusion

In this paper, we consider elliptic equation as a transformation to solve ZK equations, and multiple structures of nonlinear Rossby waves are derived, including periodic solutions of rational forms, solitary wave solutions constructed in terms of hyperbolic functions of rational forms. And application of transformation (5) to nonlinear wave equations, some of the obtained solutions have not been obtained by the sine-cosine method [17], the homogeneous balance method [18], the hyperbolic function expansion method [8,19], the Jacobi elliptic function expansion method [20,21], the nonlinear transformation method [22–24], the trial function method [25,26] and so on. So there still exists the require for further studies of nonlinear model.

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