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Exact solutions to sine-Gordon-type equations

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Abstract

In this Letter, sine-Gordon-type equations, including single sine-Gordon equation, double sine-Gordon equation and triple sine-Gordon equation, are systematically solved by Jacobi elliptic function expansion method. It is shown that different transformations for these three sine-Gordon-type equations play different roles in obtaining exact solutions, some transformations may not work for a specific sine-Gordon equation, while work for other sine-Gordon equations.

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1. Introduction

Sine-Gordon-type equations, including single sine-Gordon (SSG for short) equation

$$u_{xt} = \alpha \sin u, \tag{1}$$

double sine-Gordon (DSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u, \tag{2}$$

and triple sine-Gordon (TSG for short) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u + \gamma \sin 3u, \tag{3}$$

are widely applied in physics and engineering. For example, DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics [1–5]. Actually, SSG equation and DSG equation also arise in nonlinear optics ^{3}He spin waves and other fields. In a resonant fivefold degenerate medium, the propagation and creation of ultra-short optical pulses, the SSG and DSG models are usually used. However, in some cases, one has to consider other sine-Gordon equations. For instance, TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses [6].

Due to the wide applications of sine-Gordon-type equations, many solutions to them have been obtained in different functional forms, such as $\tan^{-1}\coth$, $\tan^{-1}\tanh$, $\tan^{-1}\mathrm{sech}$, $\tan^{-1}\mathrm{sn}$ and so on, by different methods [7–9]. In this Letter, we will show the systematical results about solutions for these three sine-Gordon-type equations. Here different transformations will be introduced to derive more types of solutions, of course, some transformations may not work for a specific sine-Gordon-type equation.

2. Solutions to SSG equation

In order to solve the sine-Gordon-type equations, certain transformations must be introduced. For example, the transformation

$$u = 2\tan^{-1} v \quad \text{or} \quad v = \tan\frac{u}{2}, \tag{4}$$

has been introduced in Refs. [7,9] to solve DSG equation.

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When the transformation (4) is considered, there are

$$\sin u = \frac{2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}} = \frac{2v}{1+v^2}, \tag{5}$$

and

$$u_{tx} = \frac{2}{1+v^2}v_{tx} - \frac{4v}{(1+v^2)^2}v_t v_x. \tag{6}$$

Combining (5) and (6) with (1), the SSG equation can be rewritten as

$$\left(1+v^2\right)v_{tx} - 2vv_t v_x - \alpha v - \alpha v^3 = 0. \tag{7}$$

Eq. (7) can be solved in the frame

$$v = v(\xi), \quad \xi = k(x-ct), \tag{8}$$

where k and c are wave number and wave speed, respectively.

Substituting (8) into (7), we have

$$k^2c\left(1+v^2\right)\frac{d^2v}{d\xi^2} - 2k^2cv\left(\frac{dv}{d\xi}\right)^2 + \alpha v + \alpha v^3 = 0, \tag{9}$$

which can be solved directly by Jacobi elliptic function expansion method [10,11]. For instance, the ansatz solution can be written as

$$v = a_0 + a_1 \operatorname{sn}\xi, \tag{10}$$

where $\operatorname{sn}\xi$ is Jacobi elliptic sine function [12–14].

The constants a_0 and a_1 can be determined by substituting (10) into (9) as

$$a_0 = 0, \qquad a_1 = \sqrt{-\frac{\alpha-(1+m^2)k^2c}{2k^2c}}, \tag{11}$$

with

$$c = \pm\frac{\alpha}{k^2(1-m^2)}, \tag{12}$$

where $0 \leqslant m \leqslant 1$ is called modulus of Jacobi elliptic functions, see [12–14].

Thus, the solution to the SSG equation is

$$u_{1S} = 2\tan^{-1}\left[\sqrt{-\frac{\alpha-(1+m^2)k^2c}{2k^2c}}\operatorname{sn}\xi\right], \quad m \neq 1. \tag{13}$$

The second transformation is introduced in the form

$$u = 2\sin^{-1}v \quad \text{or} \quad v = \sin\frac{u}{2}, \tag{14}$$

and then

$$\sin u = 2\sin\frac{u}{2}\cos\frac{u}{2} = 2v\sqrt{1-v^2}, \tag{15}$$

and

$$u_{tx} = \frac{2}{\sqrt{1-v^2}}v_{tx} + \frac{2v}{(1-v^2)\sqrt{1-v^2}}v_t v_x. \tag{16}$$

Combining (15) and (16) with (1), the SSG equation can be rewritten as

$$\left(1-v^2\right)v_{tx} + vv_t v_x - \alpha v\left(1-v^2\right)^2 = 0. \tag{17}$$

In the travelling wave frame (8), the formal solution of Eq. (17) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi. \tag{18}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = 0, \qquad a_1 = \pm 1, \qquad c = \frac{\alpha}{m^2k^2}, \tag{19}$$

or

$$a_0 = 0, \qquad a_1 = \pm m, \qquad c = \frac{\alpha}{k^2}. \tag{20}$$

Thus, we can obtain two more solutions to the SSG equation:

$$u_{2S} = \pm 2\sin^{-1}[\operatorname{sn}\xi], \tag{21}$$

and

$$u_{3S} = \pm 2\sin^{-1}[m\operatorname{sn}\xi]. \tag{22}$$

Moreover, it is known that when $m \to 1$, $\operatorname{sn}(\xi, m) \to \tanh\xi$. So we can get more kinds of solution expressed in terms of hyperbolic function,

$$u_{4S} = \pm 2\sin^{-1}[\tanh\xi], \tag{23}$$

with

$$c = \frac{\alpha}{k^2}. \tag{24}$$

Next, we introduce the third transformation

$$u = \cos^{-1}v \quad \text{or} \quad v = \cos u, \tag{25}$$

and then

$$\sin u = \sqrt{1-v^2}, \tag{26}$$

and

$$u_{tx} = -\frac{1}{\sqrt{1-v^2}}v_{tx} - \frac{v}{(1-v^2)\sqrt{1-v^2}}v_t v_x. \tag{27}$$

Combining (26) and (27) with (1), the SSG equation can be rewritten as

$$\left(1-v^2\right)v_{tx} + vv_t v_x + \alpha\left(1-v^2\right)^2 = 0. \tag{28}$$

In the travelling wave frame (8), the formal solution of Eq. (28) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi + a_2\operatorname{sn}^2\xi. \tag{29}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = \pm 1, \qquad a_1 = 0, \qquad a_2 = -\frac{2m^2k^2c}{\alpha}, \tag{30}$$

with

$$c = \frac{\alpha}{m^2k^2} \quad \text{for } a_0 = 1, \tag{31}$$

and

$$c = \frac{\alpha}{k^2} \quad \text{for } a_0 = -1, \tag{32}$$

or

$$a_0 = \frac{(1+m^2)k^2c}{\alpha}, \qquad a_1 = 0, \qquad a_2 = -\frac{2m^2k^2c}{\alpha}, \tag{33}$$

with

$$c = \pm\frac{\alpha}{(1-m^2)k^2}. \tag{34}$$

Thus, we can obtain still more solutions to the SSG equation

$$u_{5S} = \cos^{-1}\left[1 - \frac{2m^2k^2c}{\alpha}\operatorname{sn}^2\xi\right], \tag{35}$$

$$u_{6S} = \cos^{-1}\left[-1 - \frac{2m^2k^2c}{\alpha}\operatorname{sn}^2\xi\right], \tag{36}$$

and

$$u_{7S} = \cos^{-1}\left[\frac{(1+m^2)k^2c}{\alpha} - \frac{2m^2k^2c}{\alpha}\operatorname{sn}^2\xi\right]. \tag{37}$$

Moreover, when $m \to 1$, $\operatorname{sn}(\xi, m) \to \tanh\xi$, we can get more kinds of solution expressed in terms of hyperbolic function,

$$u_{8S} = \cos^{-1}\left[\pm 1 - \frac{2k^2c}{\alpha}\tanh^2\xi\right], \tag{38}$$

with

$$c = \frac{\alpha}{k^2}. \tag{39}$$

3. Solutions to DSG equation

First of all, we consider the first transformation (4), then we have

$$\sin 2u = \frac{4v(1-v^2)}{(1+v^2)^2}. \tag{40}$$

Combining (5), (6) and (40) with (2), the DSG equation can be rewritten as

$$\left(1+v^2\right)v_{tx} - 2vv_tv_x - (\alpha+2\beta)v - (\alpha-2\beta)v^3 = 0. \tag{41}$$

In the travelling wave frame (8), the formal solution of Eq. (41) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi. \tag{42}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = 0, \qquad a_1 = \pm\sqrt{-\frac{2m^2k^2c}{(\alpha-2\beta)+(1+m^2)k^2c}}, \tag{43}$$

with

$$c = \frac{2\beta(1+m^2) \pm \sqrt{(1-m^2)^2\alpha + 16m^2\beta^2}}{(1-m^2)^2k^2} \quad \text{for } m \neq 1, \tag{44}$$

and

$$c = -\frac{\alpha^2 - 4\beta^2}{8\beta k^2} \quad \text{for } m = 1. \tag{45}$$

Thus, the solutions to the DSG equation are

$$u_{1D} = \pm 2\tan^{-1}\left[\sqrt{-\frac{2m^2k^2c}{(\alpha-2\beta)+(1+m^2)k^2c}}\operatorname{sn}\xi\right], \quad m \neq 1, \tag{46}$$

and

$$u_{2D} = \pm 2\tan^{-1}\left[\sqrt{-\frac{2k^2c}{(\alpha-2\beta)+2k^2c}}\tanh\xi\right]. \tag{47}$$

When the second transformation (14) is considered, it can be easily proven that the DSG equation (2) cannot be solved directly. So, the second transformation (14) does not work for the DSG equation (2), more further transformations are needed. This case does not happen in the third transformation (25), where we have

$$\sin 2u = 2v\sqrt{1-v^2}. \tag{48}$$

Combining (26), (27) and (48) with (2), the DSG equation can be rewritten as

$$\left(1-v^2\right)v_{tx} + vv_tv_x + \alpha\left(1-v^2\right)^2 + 2\beta v\left(1-v^2\right)^2 = 0. \tag{49}$$

In the travelling wave frame (8), the formal solution of Eq. (49) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\operatorname{sn}\xi. \tag{50}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = -\frac{\alpha}{4\beta}, \qquad a_1 = \pm\sqrt{-\frac{m^2k^2c}{2\beta}}, \tag{51}$$

with

$$c = -\frac{\alpha^2 + 16\beta^2}{4(1+m^2)k^2\beta}. \tag{52}$$

Thus, another solution to the DSG equation is

$$u_{3D} = \cos^{-1}\left[-\frac{\alpha}{4\beta} \pm \sqrt{-\frac{m^2k^2c}{2\beta}}\operatorname{sn}\xi\right]. \tag{53}$$

When $m \to 1$, we have

$$u_{4D} = \cos^{-1}\left[-\frac{\alpha}{4\beta} \pm \sqrt{-\frac{k^2c}{2\beta}}\tanh\xi\right], \tag{54}$$

with

$$c = -\frac{\alpha^2 + 16\beta^2}{8k^2\beta}. \tag{55}$$

4. Solutions to TSG equation

First of all, we consider the first transformation (4), then we have

$$\sin 3u = \frac{6v}{1+v^2} - 4\left(\frac{2v}{1+v^2}\right)^3. \tag{56}$$

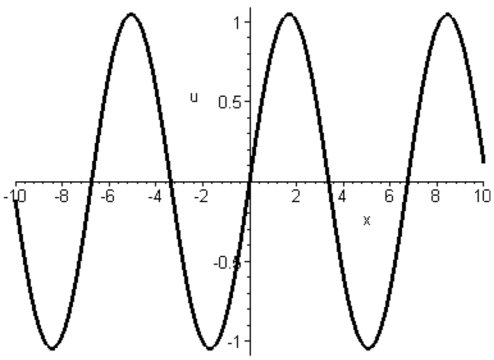

Fig. 1. Graphical presentation for solution $u = 2\sin^{-1}[m\,\mathrm{sn}(x,m)]$ for $m = 0.5$.

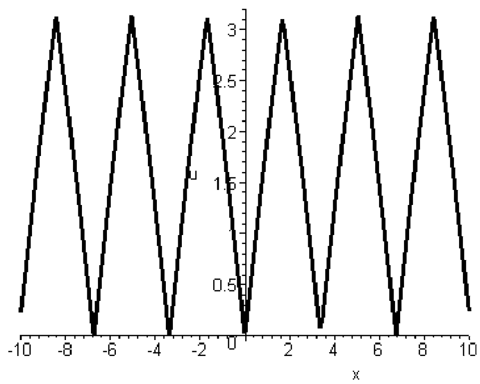

Fig. 2. Graphical presentation for solution $u = \cos^{-1}[1 - \frac{2m^2k^2c}{\alpha}\,\mathrm{sn}^2(x,m)]$ for $m = 0.5$, $\alpha = 1$.

Combining (5), (6), (40) and (56) with (3), the TSG equation can be rewritten as

$$\left(1+v^2\right)v_{tx} - 2vv_tv_x - (\alpha+2\beta+3\gamma)v - 2(\alpha-5\gamma)v^3 - (\alpha-2\beta+3\gamma)v^5 = 0. \tag{57}$$

In the travelling wave frame (8), the formal solution of Eq. (57) by the Jacobi elliptic function expansion method [10, 11] can be written as

$$v = a_0 + a_1\,\mathrm{sn}\,\xi. \tag{58}$$

Similarly, the expansion coefficients can be determined as

$$a_0 = 0, \qquad a_1 = \pm\sqrt{\frac{5\gamma}{(1-m^2)k^2c}}, \quad m \neq 1, \tag{59}$$

here the solution to the TSG equation (3) is

$$u_{1\mathrm{T}} = \pm 2\tan^{-1}\left[\sqrt{\frac{5\gamma}{(1-m^2)k^2c}}\,\mathrm{sn}\,\xi\right], \quad m \neq 1. \tag{60}$$

Unfortunately, the second transformation (14) and the third transformation (25) cannot be applied to solve the TSG equation (3) directly, i.e., the second transformation (14) and the third transformation (25) do not work for the TSG equation (3), more further transformations are needed.

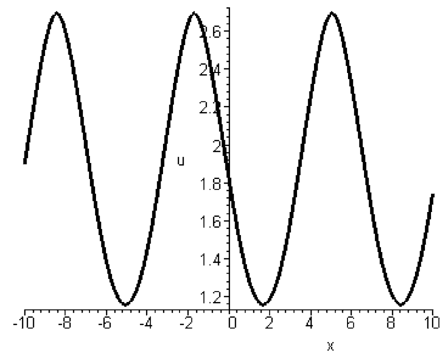

Fig. 3. Graphical presentation for solution $u = \cos^{-1}[-\frac{\alpha}{4\beta} + \sqrt{-\frac{m^2k^2c}{2\beta}} \times \mathrm{sn}(x,m)]$ for $m = 0.5$, $\alpha = 1$, $\beta = 1$.

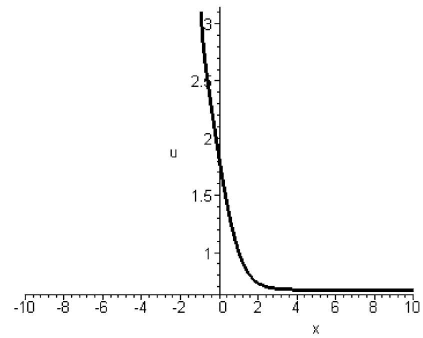

Fig. 4. Graphical presentation for solution $u = \cos^{-1}[-\frac{\alpha}{4\beta} + \sqrt{-\frac{k^2c}{2\beta}}\tanh(x)]$ for $\alpha = 1$, $\beta = 1$.

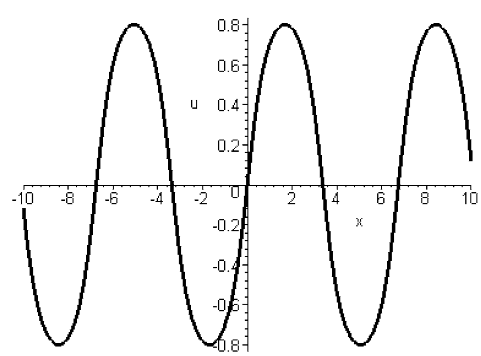

Fig. 5. Graphical presentation for solution $u = 2\tan^{-1}[\sqrt{\frac{5\gamma}{(1-m^2)k^2c}}\,\mathrm{sn}(x,m)]$ for $m = 0.5$, $\gamma = 1$.

5. Conclusion

In this Letter, the sine-Gordon-type equations, including single sine-Gordon equation, double sine-Gordon equation and triple sine-Gordon equation, are systematically solved by Jacobi elliptic function expansion method. Here some new solutions have not been reported in the literature, for example, for the DSG equation (2), $u_{3\mathrm{D}}$ and $u_{4\mathrm{D}}$ have not been reported in the related works [3,4,7,9]. It is shown that different transformations for these three sine-Gordon-type equations play different roles in obtaining exact solutions, some transformations may not work for a specific sine-Gordon equation, while work for other sine-Gordon equations. Of course, there are still more efforts needed to explore what kinds of transformations are more suitable to solve these sine-Gordon-type equations, especially for the higher multiple sine-Gordon equations. Because different transformations result in different partial balances for these sine-Gordon-type equations, which will lead to different expansion truncations in the Jacobi elliptic function expansion method. Finally, these will result in different solutions of the sine-Gordon-type equations. Details can be found in the above graphical representations of the solutions. From these graphical representations, we can see that the solutions take different behaviors when different transformations (see Fig. 1 and Fig. 2) and different parameters (see Fig. 3 and Fig. 4) are chosen.

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