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A Nonlinear Coupled Soil Moisture-Vegetation Model

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ABSTRACT

Based on the physical analysis that the soil moisture and vegetation depend mainly on the precipitation and evaporation as well as the growth, decay and consumption of vegetation a nonlinear dynamic coupled system of soil moisture-vegetation is established. Using this model, the stabilities of the steady states of vegetation are analyzed. This paper focuses on the research of the vegetation catastrophe point which represents the transition between aridness and wetness to a great extent. It is shown that the catastrophe point of steady states of vegetation depends mainly on the rainfall P and saturation value v_0, which is selected to balance the growth and decay of vegetation. In addition, when the consumption of vegetation remains constant, the analytic solution of the vegetation equation is obtained.

Key words: vegetation, steady states, equilibrium points, catastrophe point, cusp catastrophe

1. Introduction

We can see from El Niño events that the ocean plays an important role in global climate variation. In the same way as the action of the ocean on the climate changes, the land surface also has a great effect on the climate variation. In the research of aridness, the soil moisture and vegetation in the land surface process are two very important variables. Physical analysis shows that the variations of soil moisture and vegetation are both concerned with the rainfall; the soil moisture and vegetation increase and decrease simultaneously due to their interaction (Holdridge, 1974; Lieth, 1975; Ma and Osamu, 2002). Eagleson (1978a–f, 1982, 2002) presented a statistical-dynamic model on the average water balance during the vegetation growing season in terms of three variables: average soil moisture, canopy cover and canopy conductance. This model has provided insight into the physical basis for the role played by water in the growth of vegetation communities. Rodriguez-Iturbe and his collaborators (Rodriguez-Iturbe et al., 2001; Porporato et al., 2001; Laio et al., 2001a; Laio et al., 2001b; Guswa et al., 2002; Fernandez-Illescas et al., 2004) have deeply studied plants in water controlled ecosystems. Zeng and Neelin (1999, 2000), Zeng et al. (2002) have analyzed the relationship between the vegetation and rainfall and the interaction of vegetation and climate, and they established a nonlinear model on the coupled vegetation-atmosphere system. However, they did not analyze the catastrophe point of vegetation which represents the climate change; they also did not discuss the quick and slow variables in the coupled system.

In this paper, the major factors which determine the changes of soil moisture and vegetation are analyzed, and a nonlinear dynamic system on the coupled soil moisture and vegetation is established. Using this system, the catastrophe point of vegetation and the evolutions of the soil moisture and vegetation are studied. It is shown that the catastrophe point of vegetation represents the transition between the climatic aridness and wetness to a great extent.

2. Physical analysis and mathematical model of the variation of soil moisture

Assuming that the soil moisture is m and disregarding the runoff and irrigation, the variation of m with time is dependent mainly on the precipitation rate P and evaporation rate E (Zeng et al., 1994), namely,

$$\frac{dm}{dt} = P - E\,. \tag{1}$$

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The research by Walker and Rowntree (1977) shows that the soil moisture and rainfall are variant simultaneously. Yeh et al. (1984) simulated the effect of great increases in soil moisture on the climate variation by a simple short-term climatic model and showed that the increases in the precipitation and evaporation are bound to follow the increases in the soil moisture.

The above analysis implies that P and E are directly proportional to m, but as a whole in the arid and semi-arid regions of the northern part of China, the precipitation times are far shorter than those of evaporation. So we assume that

$$P - E \propto -m \,. \tag{2}$$

In fact the variation of soil moisture depends also on the vegetation cover v. Generally speaking, the higher the precipitation rate, the greater the vegetation cover and the lower the evaporation rate. So the following assumption can be made

$$P - E \propto v \,. \tag{3}$$

Thus, by means of Eqs. (2) and (3), Eq. (1) may be rewritten as

$$\frac{dm}{dt} = -a_1 m + b_1 v \,, \tag{4}$$

where $a_1 > 0$ and $b_1 > 0$, and a_1 is known as the linear decaying rate of the soil moisture, while b_1 represents the coupled action of vegetation on the variation of soil moisture.

3. Physical analysis and mathematical model of the variation of vegetation cover

The variation of vegetation cover v with time depends mainly on the growth rate G, decay rate D and consumption rate C (Zeng et al., 1994), namely,

$$\frac{dv}{dt} = G - D - C \,. \tag{5}$$

Firstly, similar to the fact that increases in the precipitation rate and evaporation rate are bound to follow increases in the soil moisture, for a given air temperature and sunlight amount, G and D both increase with increases in vegetation cover v. However, because of natural factors (sunlight, rodents and so on) v cannot increase infinitely and reaches only a saturated value after which the vegetation amount decays. Therefore, we assume that

$$G - D \propto v\left(1 - \frac{v}{v_0}\right) \tag{6}$$

which implies that $G \leqslant D$ at $v \geqslant v_0$.

Secondly, $G - D$ is of great importance to the soil moisture m. Generally speaking, the greater the soil moisture, the greater the growth rate and the smaller the decay rate of vegetation. So we have

$$G - D \propto m \,. \tag{7}$$

As for the consumption of vegetation, it includes artificial felling and grazing. Usually, the consumption rate depends indirectly on the vegetation cover. However, there is an approximate threshold v_c for vegetation. The consumption is smaller at $v < v_c$ and larger at $v > v_c$. We take the form for C suggested by Ludwig et al. (1978)

$$C = \frac{bv^2}{a^2 + v^2} \,, \tag{8}$$

where a and b are positive constants which have determinate meanings. The variations of the consumption rate with v are illustrated in Fig. 1. We see from Fig. 1 and Eq. (8) that

$$v_c = \frac{a}{\sqrt{3}} \text{ (it satisfies } \frac{\partial^2 C}{\partial v^2} = 0) \,,$$

$$b = \lim_{v \to \infty} C \,.$$

By making use of Eqs. (6), (7) and (8), Eq. (5) may be written as

$$\frac{dv}{dt} = a_2 v\left(1 - \frac{v}{v_0}\right) + b_2 m - \frac{bv^2}{a^2 + v^2} \,, \tag{9}$$

where $a_2 > 0$ and $b_2 > 0$, and a_2 represents the linear growth rate of vegetation, while b_2 denotes the coupled action of soil moisture on the variation of vegetation.

4. A nonlinear coupled soil moisture-vegetation model

Combining Eqs. (4) and (9), a simple nonlinear coupled soil moisture-vegetation model is given by

$$\frac{dm}{dt} = -a_1 m + b_1 v \,, \tag{10a}$$

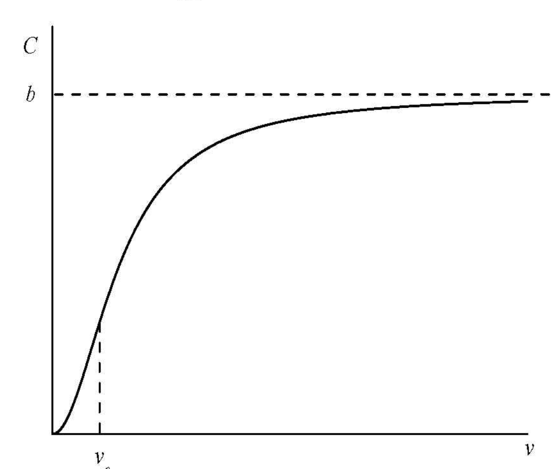

Fig. 1. The variation of vegetation consumption rate with the vegetation over.

$$\frac{dv}{dt} = a_2 v \left(1 - \frac{v}{v_0}\right) + b_2 m - \frac{bv^2}{a^2 + v^2} . \tag{10b}$$

We see from Eq. (10) that m and v increase or decrease simultaneously by their interaction.

The research on nonlinear dynamic systems is divided into two parts. In the first part, the quick or slow variations of m and v are studied, and focus is laid on the catastrophe behavior of the dynamic system. In the second part, the research focuses on the evolutions of soil moisture and vegetation under certain conditions.

5. The catastrophe point in the nonlinear coupled system

We can see from Eq. (10) that there exists a damping to changes in m. Hence, comparatively speaking, m is called the fast variable and v is called the slow variable which dominates the behavior of system. For this reason, we introduce the small parameter ε ($0 < \varepsilon \ll 1$) and rewrite Eq. (10) as

$$\varepsilon \frac{dm}{dt} = -a_1 m + b_1 v , \tag{11a}$$

$$\frac{dv}{dt} = a_2 v \left(1 - \frac{v}{v_0}\right) + b_2 m - \frac{bv^2}{a^2 + v^2} . \tag{11b}$$

Setting $\varepsilon = 0$, one has

$$m = \frac{b_1}{a_1} v . \tag{12}$$

Substituting Eq. (12) into Eq. (11b) yields

$$\frac{dv}{dt} = a_2 v \left(1 - \frac{v}{v_0}\right) + \frac{b_1 b_2}{a_1} v - \frac{bv^2}{a^2 + v^2} \tag{13}$$

which is the evolution equation for the slow variable v. In Eq. (13) the first term on the right hand represents the net effect between the growth and decay of vegetation cover, where a_2 is the linear growth rate in v, and it mainly depends on the precipitation P. The second term represents the coupled effect between the soil moisture and vegetation cover, and it is also related to the precipitation. The third term represents the consumption rate of vegetation cover.

Introducing the following non-dimensional quantities

$$v^* = \frac{v}{a}, \quad p = \frac{aa_2}{b} ,$$
$$q = \frac{ab_1 b_2}{a_1 b}, \quad v_0^* = \frac{v_0}{a} ,$$
$$t^* = \frac{bt}{a} , \tag{14}$$

then Eq. (13) can be rewritten as

$$\frac{dv}{dt} = pv \left(1 - \frac{v}{v_0}\right) + qv - \frac{v^2}{1 + v^2} , \tag{15}$$

where the star symbol is omitted. In Eq. (15), p is a non-dimensional precipitation and q is also related to the precipitation, which denotes the coupling coefficient of soil moisture and vegetation cover. The model (15) is similar to the results given by Murray (2000) when $q = 0$.

Suppose that

$$q = \alpha p ,$$
$$g(v) = p \left(1 - \frac{v}{v_0}\right) + q = p \left(1 + \alpha - \frac{v}{v_0}\right) ,$$
$$h(v) = \frac{v^2}{1 + v^2} , \tag{16}$$

where α is a coefficient, then Eq. (15) may be rewritten as

$$\frac{dv}{dt} = v[g(v) - h(v)] . \tag{17}$$

The steady states (equilibria) of vegetation satisfy $v = 0$ (trivial) or

$$g(v) = h(v) \tag{18}$$

which is the set of intersections of the straight line $g(v)$ and the curve $h(v)$.

The changes of the straight line $g(v)$ and the curve $h(v)$ with v are shown in Fig. 2. The intersection points of the straight line $g(v)$ with the abscissa and the ordinate are $(1 + \alpha)v_0$ and $(1 + \alpha)p$, respectively.

Eq. (18) can be reduced to a cubic algebraic equation

$$v^3 - (1+\alpha)v_0 v^2 + \left(1 + \frac{v_0}{p}\right) v - (1+\alpha)v_0 = 0 . \tag{19}$$

In Fig. 2, for the straight line (1), there is only one intersection point P (stable) with the curve $h(v)$, which implies that there is only one real root in Eq. (19). For the straight line (2), there are three intersection points Q (stable), R (unstable) and S (stable) with the curve $h(v)$, with abscissa v_1, v_2 and v_3, respectively, which implies that there are three real roots in Eq. (19). For the straight line (3), there are two intersection points U (stable) and T (stable), which coincid with Q and R. For the straight line (4), there is only one intersection point V with the curve $h(v)$. Therefore, the steady states of vegetation cover jump from one to three as the parameter p changes. This is just a catastrophe of the vegetation cover; T is the catastrophe point in which not only $g(v)$ and $h(v)$ intersect, but also $g(v)$ is the tangent line of $h(v)$ at this point. It satisfies

$$g(v) = h(v), \quad \frac{\partial g}{\partial v} = \frac{\partial h}{\partial v} , \tag{20}$$

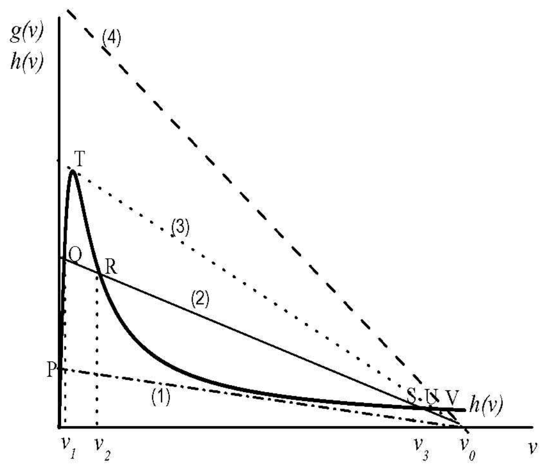

Fig. 2. Solutions for the steady states of the vegetation cover represented by $g(v) = h(v)$ [Eq. (18)] for straight line $g(v)$ and curve $h(v)$. For straight lines are given (1)–(4) for different values of the parameters. the intersection points are labeled as P, Q, R, S, T, U, and V.

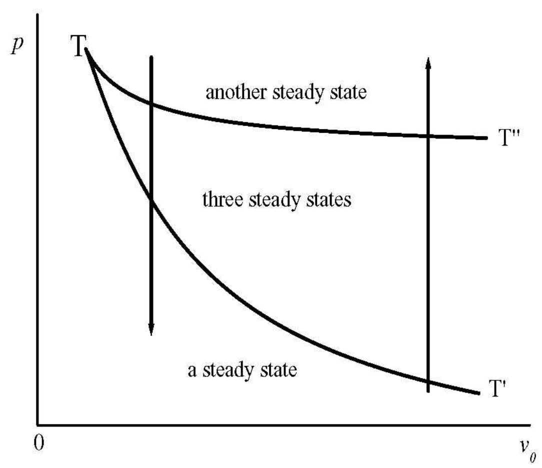

Fig. 3. The cusp catastrophe of the nonlinear coupled system represented by Eq. (22).

namely,

$$p\left(1+\alpha-\frac{v}{v_0}\right)=\frac{v}{1+v^2}\,,-\frac{p}{v_0}=\frac{1-v^2}{(1+v^2)^2}\,. \qquad (21)$$

Hence the parametric equations of the catastrophe point are given by

$$(1+\alpha)p=\frac{2v^3}{(1+v^2)^2}\,,\quad \alpha v_0=\frac{2v^3}{v^2-1}\,. \qquad (22)$$

In the parameter plane (v_0, p), Eq. (22) represents two critical curves which divide the plane into two regions where there is one or three steady states. The two curves meet at a cusp where p and v_0 no longer vary with v, that is,

$$\frac{dp}{dv}=\frac{dv_0}{dv}=0$$

at

$$v=\sqrt{3}$$

and

$$(v_0, p)=(3\sqrt{3}, \frac{3\sqrt{3}}{8})$$

is the location of catastrophe point T which is shown in Fig. 3. This is known as a cusp catastrophe.

In Fig. 3, one steady state lies below the curve TT', where v is smaller, just as point P in Fig. 2, which corresponds to a state of aridness; one steady state lies above the curve TT'', where v is larger, just as point V in Fig. 2, which corresponds to a state of wetness. Thus, along the arrowed path in Fig. 3, there is a jump from the arid state up to the wet state as p increases; otherwise, there is a jump from the wet state down to the arid state as p decreases.

6. The analytic solution of the vegetation equation for a constant consumption rate

In this section, if we consider a constant consumption rate, then Eq. (10) can be rewritten as

$$\frac{dm}{dt}=-a_1m+b_1v\,, \qquad (23a)$$

$$\frac{dv}{dt}=a_2v\left(1-\frac{v}{v_0}\right)+b_2m-C\,, \qquad (23b)$$

where the consumption rate of vegetation is the constant C. By elimination, we obtain

$$\frac{d^2v}{dt^2}=\left[(a_2-a_1)-\frac{2a_2}{v_0}v\right]\frac{dv}{dt} + (a_1a_2+b_1b_2)v-\frac{a_1a_2}{v_0}v^2-a_1C\,. \qquad (24)$$

In Eq. (24), dv/dt and d^2v/dt^2 can be seen as the velocity and acceleration of vegetation growth, respectively. But, in view of the differential equation, the term including dv/dt represents a damping. Disregarding the damping term, then Eq. (24) reduces to

$$\frac{d^2v}{dt^2}=(a_1a_2+b_1b_2)v-\frac{a_1a_2}{v_0}v^2-a_1C\,. \qquad (25)$$

Multiplying Eq. (25) by dv/dt and integrating once yields

$$\left(\frac{dv}{dt}\right)^2=-\frac{a_1a_2}{3v_0}\left[v^3-\frac{3v_0(a_1a_2+b_1b_2)}{a_1a_2}v^2 +\frac{6v_0}{a_2}Cv+C_1\right]\,, \qquad (26)$$

where C_1 is an integration constant.

Assuming that the cubic polynomial of v in the square bracket on the right hand of Eq. (26) has three real zeroes, $v^{(1)}, v^{(2)}$ and $v^{(3)} (v^{(1)} \geqslant v^{(2)} \geqslant v^{(3)})$, then the solution of Eq. (26) is (Liu and Liu, 2000)

$$v = v^{(2)} + (v^{(1)} - v^{(2)}) \times \mathrm{cn}^2 \left(\sqrt{\frac{a_1 a_2 (v^{(1)} - v^{(3)})}{12 v_0}} t, k \right) , \qquad (27)$$

where $\mathrm{cn}(x, k)$ is the Jacobi elliptic cosine function (Liu and Liu, 2000) with the modulus k defined as

$$k = \sqrt{\frac{v^{(1)} - v^{(2)}}{v^{(1)} - v^{(3)}}} \qquad (28)$$

Eq. (27) means that when the consumption rate of vegetation is a constant C, then the vegetation cover, whose values imply the state of climatic aridness or wetness, is a periodic variation, whose period depends on v_0, a_1, a_2 and so on.

7. Conclusion and discussion

In a nonlinear coupled soil moisture-vegetation system, we have demonstrated the effect of precipitation on climate catastrophe points which represent the transition between the arid state the aridness and the wet state to a great extent. The results show that the catastrophe points of steady states of vegetation are determined mainly by the rainfall P and saturated value v_0, which keeps the balance between the growth and decay of vegetation. In addition, if the consumption rate of vegetation remains constant, then a periodic solution which represents the climate change with a certain period is obtained for the vegetation cover.

In fact, similar to other nonlinear systems (Zeng and Neelin, 1999, 2000; Zeng et al., 2002; Eagleson, 1978a–f, 1982; Rodrguez-Iturbe, 2001; Guswa et al., 2002; Fernandez-Illescas and Rodriguez-Iturbe, 2004) in which the vegetation cover is also included, this nonlinear coupled soil moisture-vegetation-precipitation model has demonstrated the role of the vegetation feedback on climate variability. A positive vegetation feedback, which implies that the rainfall is heavy, will lead to a humid or semi-humid climate state; while a negative vegetation feedback, which denotes that the rainfall is scarce, will result in an arid or semi-arid climate state. And the catastrophe point between thses two climate states can happen with an increasing or decreasing rainfall. As a result, a good climate model must take the vegetation into account. Of course, the variation of the vegetation with time should be accurate, though the present observational evidence is not sufficient to give us a more precise constraint on the vegetation. In addition, in order to investigate the influence of different factors on the vegetation, the nonlinear equation on the vegetation should also be solved more accurately.

Our model used in this study is simple, especially the physical analysis and model of the vegetation. So the interaction between vegetation and climate as well as a good climate model with the vegetation need to be investigated further.

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