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Breather Lattice Solutions to Negative mKdV Equation*

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Abstract *In this paper, dependent and independent variable transformations are introduced to solve the negative mKdV equation systematically by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different kinds of solutions can be obtained to the negative mKdV equation, including breather lattice solution and periodic wave solution.*

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Key words: Jacobian elliptic function, the negative mKdV equation, periodic wave solution, breather lattice solution

1 Introduction

Among the soliton bearing nonlinear equations, the modified Korteweg-de Vries (mKdV) equation is of special interest.[1,2] For it possesses rich solutions, such as solitary solutions,[1−4] periodic solutions,[3−6] breather solution,[1,2,7,8] breather lattice solutions.[7,8] A particularly interesting type of solution is the so called breather-type of solutions, usually this kind of solutions are unavailable and such solutions have to be solved numerically.[7] In some cases, however, the analytical expressions in closed form can be found, such as the breather lattice solution for the sine-Gordon equation[9] and for mKdV eqaution.[7,8]

In the Refs. [7–9], Kevrekidis and his coauthors have applied some ansatzs to obtain the breather lattice solutions to the mKdV equation and the sine-Gordon equation. The aim of present paper is to present the breather lattice solutions of the negative mKdV equation in a systematical way. Based on the introduced transformations, we will show systematical results about these breather-type solutions for the negative mKdV equation by using the knowledge of elliptic equation and Jacobian elliptic functions.[3,4,10−12]

2 Breather Lattice Solutions to Negative mKdV Equation

The negative mKdV equation reads[7,8]

$$u_t - 6u^2 u_x + u_{xxx} = 0\,. \tag{1}$$

In order to derive the breather-type solutions to the negative mKdV equation (1), first of all, we introduce a dependent variable transformation

$$u = 2\frac{\partial}{\partial x}\tanh^{-1}\phi\,, \tag{2}$$

and then ϕ satisfies the equation[7,8]

$$(1-\phi^2)(\phi_t + \phi_{xxx}) - 6\phi_x(\phi_x^2 - \phi\phi_{xx}) = 0\,, \tag{3}$$

which can be taken as another form of the negative mKdV equation (1).

Next, we introduce independent variable transformation

$$\xi = ax + bt + \xi_0, \quad \eta = cx + dt + \eta_0\,, \tag{4}$$

where ξ_0 and η_0 are two constants.

Considering the transformation (4), Eq. (3) can be rewritten as

$$\begin{aligned}(1-\phi^2)[(b\phi_\xi + d\phi_\eta) + (a^3\phi_{\xi\xi\xi} + 3a^2c\phi_{\xi\xi\eta} + 3ac^2\phi_{\xi\eta\eta} + c^3\phi_{\eta\eta\eta})]\\ - 6(a\phi_\xi + c\phi_\eta)[(a\phi_\xi + c\phi_\eta)^2 - \phi(a^2\phi_{\xi\xi} + 2ac\phi_{\xi\eta} + c^2\phi_{\eta\eta})] = 0\,.\end{aligned} \tag{5}$$

Compared to the transformation given in Refs. [1,2], transformation (4) has less constraints, of course, this will let us have more different types of solutions to the negative mKdV equation (1).

Inspired by the transformation given in Ref. [2] and the results in Refs. [7,8], we choose dependent variable transformation

$$\phi = \alpha U(\xi)V(\eta)\,, \tag{6}$$

where α is a constant amplitude to be determined, U and V satisfy the following elliptic equation

$$U_\xi^2 = -n^2U^4 + p_1U^2 + q_1\,,$$

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$$V_\eta^2 = -\beta n^2 V^4 + p_2 V^2 + q_2\,, \tag{7}$$

where β, n^2, p_1, p_2, q_1, and q_2 are determined constants. Here one point must be stressed is that the introduction of β will let us have more choices to obtain different kinds of solutions to the negative mKdV equation.

Substituting (6) and (7) into (5) yields the following algebraic equations

$$b + p_1 a^3 + 3p_2 a c^2 = 0\,, \tag{8a}$$
$$\beta n^2 c^2 + q_1 \alpha^2 a^2 = 0\,, \tag{8b}$$
$$q_2 \alpha^2 c^2 + n^2 a^2 = 0\,, \tag{8c}$$
$$d + 3p_1 a^2 c + p_2 c^3 = 0\,, \tag{8d}$$

from which we can determine

$$\alpha^4 = \frac{\beta n^4}{q_1 q_2}, \quad \frac{a^2}{c^2} = -\frac{\beta n^2}{q_1 \alpha^2} = -\frac{q_2 \alpha^2}{n^2}\,,$$
$$b = -a(p_1 a^2 + 3p_2 c^2), \quad d = -c(3p_1 a^2 + p_2 c^2)\,. \tag{9}$$

From (9), it is obvious that the determined constants in (7) must satisfy the following constraints

$$\frac{\beta}{q_1 q_2} > 0, \quad \frac{q_2}{n^2} < 0, \quad \frac{\beta n^2}{q_1} < 0\,, \tag{10}$$

this implies that not all combinations of Jacobi elliptic functions are solutions to the negative mKdV equation (1) under the above mentioned transformations, only the combination of a couple of the Jacobi elliptic functions satisfies the constraint (10), it can be a solution to the negative mKdV eqaution (1). Actually, there exist only 28 these kinds of combinations, we will address them in details.

Case 1 When $U = \mathrm{dn}(\xi, k)$ and $V = \mathrm{dn}(\eta, m)$, where $\mathrm{dn}(\xi, k)$ and $\mathrm{dn}(\eta, m)$ are the Jacobi elliptic function of the third kind, and k and m are their modulus.[10−12] Then from (7), we have

$$n^2 = 1, \quad p_1 = 2 - k^2, \quad q_1 = -(1 - k^2)\,,$$
$$\beta n^2 = 1, \quad p_2 = 2 - m^2, \quad q_2 = -(1 - m^2)\,. \tag{11}$$

Substituting (11) into (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1 - m^2}{1 - k^2}}\,,$$
$$\frac{b}{a} = -[a^2(2 - k^2) + 3c^2(2 - m^2)]\,,$$
$$\frac{d}{c} = -[3a^2(2 - k^2) + c^2(2 - m^2)]\,,$$
$$\alpha = \pm\Big[\frac{1}{(1 - k^2)(1 - m^2)}\Big]^{1/4}\,, \tag{12}$$

then the solution is

$$\phi_1 = \pm\Big[\frac{1}{(1 - k^2)(1 - m^2)}\Big]^{1/4}[\mathrm{dn}(\xi, k)\mathrm{dn}(\eta, m)]\,, \tag{13}$$

which is a kind of breather lattice solution given in Ref. [8].

When $k \to 0$ (or $m \to 0$), the breather lattice solution (13) turns to be a periodic wave solution

$$\phi_{1'} = \pm\Big(\frac{1}{1 - m^2}\Big)^{1/4}[\mathrm{dn}(\eta, m)]\,. \tag{14}$$

Case 2 When $U = \mathrm{sn}(\xi, k)$ and $V = \mathrm{sn}(\eta, m)$, where $\mathrm{sn}(\eta, m)$ is the Jacobi sine elliptic function.[10−12] And then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1 + k^2) + 3c^2(1 + m^2)]\,,$$
$$\frac{d}{c} = [3a^2(1 + k^2) + c^2(1 + m^2)]\,,$$
$$\alpha = \pm\sqrt{km}\,, \tag{15}$$

then the solution is

$$\phi_2 = \pm\sqrt{km}\,[\mathrm{sn}(\xi, k)\mathrm{sn}(\eta, m)]\,, \tag{16}$$

which is another kind of breather lattice solution given in Ref. [8].

Besides the above two kinds of breather lattice solutions, there still exist 26 kinds of breather lattice solutions that have not been reported in the literature, next we will show their details.

Case 3 When $U = \mathrm{sn}(\xi, k)$ and $V = \mathrm{cd}(\eta, m) = \mathrm{cn}(\eta, m)/\mathrm{dn}(\eta, m)$, where $\mathrm{cn}(\eta, m)$ is the Jacobi cosine elliptic function.[10−12] And then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1 + k^2) + 3c^2(1 + m^2)]\,,$$
$$\frac{d}{c} = [3a^2(1 + k^2) + c^2(1 + m^2)]\,,$$
$$\alpha = \pm\sqrt{km}\,, \tag{17}$$

and the breather lattice solution is

$$\phi_3 = \pm\sqrt{km}\,[\mathrm{sn}(\xi, k)\mathrm{cd}(\eta, m)]\,. \tag{18}$$

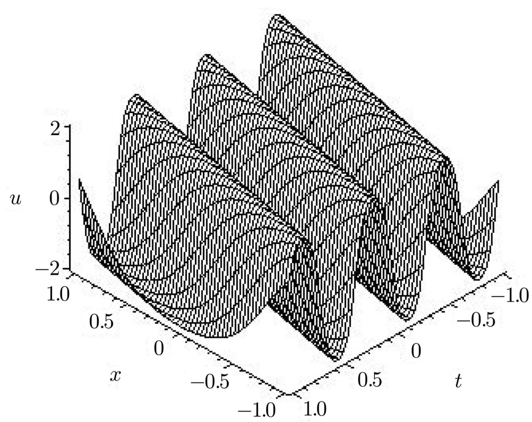

Fig. 1 The space-time evolution of the breather lattice solution of Eqs. (17) and (18). $a = 1$, $c = 1$, $m = 0.5$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = 5$, $d = 5$, $k = 0.5$, and $\alpha = 1/2$.

Figure 1 shows the evolution of the breather lattice solution with the periodic characteristics in both spatial and temporal directions, the profiles in both spatial and temporal directions are smooth.

Case 4 When $U = \mathrm{cd}(\xi, k)$ and $V = \mathrm{cd}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)],$$
$$\alpha = \pm\sqrt{km}, \tag{19}$$

then the breather lattice solution is

$$\phi_4 = \pm\sqrt{km}[\mathrm{cd}(\xi, k)\mathrm{cd}(\eta, m)]. \tag{20}$$

Case 5 When $U = \mathrm{ns}(\xi, k) = 1/\mathrm{sn}(\xi, k)$ and $V = \mathrm{ns}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)],$$
$$\alpha = \pm\frac{1}{\sqrt{km}}, \tag{21}$$

then the breather lattice solution is

$$\phi_5 = \pm\frac{1}{\sqrt{km}}[\mathrm{ns}(\xi, k)\mathrm{ns}(\eta, m)]. \tag{22}$$

Case 6 When $U = \mathrm{ns}(\xi, k)$ and $V = \mathrm{dc}(\eta, m) = \mathrm{dn}(\eta, m)/\mathrm{cn}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)],$$
$$\alpha = \pm\frac{1}{\sqrt{km}}, \tag{23}$$

then the breather lattice solution is

$$\phi_6 = \pm\frac{1}{\sqrt{km}}[\mathrm{ns}(\xi, k)\mathrm{dc}(\eta, m)]. \tag{24}$$

Case 7 When $U = \mathrm{dc}(\xi, k)$ and $V = \mathrm{dc}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)],$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)],$$
$$\alpha = \pm\frac{1}{\sqrt{km}}, \tag{25}$$

then the breather lattice solution is

$$\phi_7 = \pm\frac{1}{\sqrt{km}}[\mathrm{dc}(\xi, k)\mathrm{dc}(\eta, m)]. \tag{26}$$

Case 8 When $U = \mathrm{dn}(\xi, k)$ and $V = \mathrm{nd}(\eta, m) = 1/\mathrm{dn}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)],$$
$$\alpha = \pm\Big[\frac{1-m^2}{1-k^2}\Big]^{1/4}, \tag{27}$$

then the breather lattice solution is

$$\phi_8 = \pm\Big[\frac{1-m^2}{1-k^2}\Big]^{1/4}[\mathrm{dn}(\xi, k)\mathrm{nd}(\eta, m)]. \tag{28}$$

Case 9 When $U = \mathrm{nd}(\xi, k)$ and $V = \mathrm{nd}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)],$$
$$\alpha = \pm[(1-m^2)(1-k^2)]^{1/4}, \tag{29}$$

then the breather lattice solution is

$$\phi_9 = \pm[(1-m^2)(1-k^2)]^{1/4}[\mathrm{nd}(\xi, k)\mathrm{nd}(\eta, m)]. \tag{30}$$

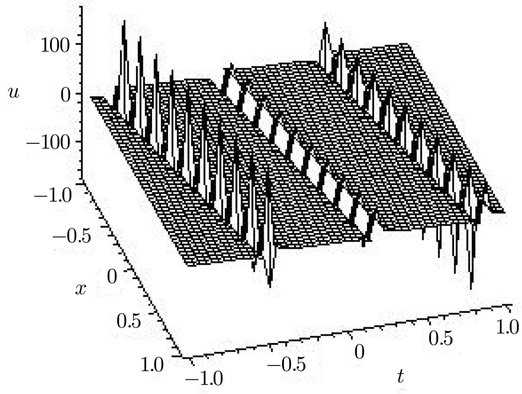

Fig. 2 The space-time evolution of the breather lattice solution of Eqs. (29) and (30). $a = 1$, $c = 1$, $m = \sqrt{3}/2$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = -5$, $d = -5$, $k = \sqrt{3}/2$, and $\alpha = 1/2$.

It is obvious that Fig. 2 describes a different kind of breather lattice solution from that given in Fig. 1. Compared to Fig. 1, the profiles in Fig. 2 for both spatial and temporal directions are intermittent. At the same time, the periodic characteristics of Fig. 2 in the spatial direction is quite different from that in the temporal direction.

Case 10 When $U = \mathrm{sc}(\xi, k) = \mathrm{sn}(\xi, k)/\mathrm{cn}(\xi, k)$ and $V = \mathrm{sc}(\eta, m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)],$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)],$$
$$\alpha = \pm[(1-m^2)(1-k^2)]^{1/4}, \tag{31}$$

then the breather lattice solution is

$$\phi_{10} = \pm[(1-m^2)(1-k^2)]^{1/4}[\mathrm{sc}(\xi, k)\mathrm{sc}(\eta, m)]. \tag{32}$$

Case 11 When $U = \mathrm{sc}(\xi,k)$ and $V = \mathrm{cs}(\eta,m) = \mathrm{cn}(\eta,m)/\mathrm{sn}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)]\,,$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)]\,,$$
$$\alpha = \pm\Big[\frac{1-k^2}{1-m^2}\Big]^{1/4}, \tag{33}$$

then the breather lattice solution is

$$\phi_{11} = \pm\Big[\frac{1-k^2}{1-m^2}\Big]^{1/4}[\mathrm{sc}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{34}$$

Case 12 When $U = \mathrm{cs}(\xi,k)$ and $V = \mathrm{cs}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{1-k^2}}, \quad \frac{b}{a} = -[a^2(2-k^2) + 3c^2(2-m^2)]\,,$$
$$\frac{d}{c} = -[3a^2(2-k^2) + c^2(2-m^2)]\,,$$
$$\alpha = \pm\Big[\frac{1}{(1-k^2)(1-m^2)}\Big]^{1/4}, \tag{35}$$

then the breather lattice solution is

$$\phi_{12} = \pm\Big[\frac{1}{(1-k^2)(1-m^2)}\Big]^{1/4}[\mathrm{cs}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{36}$$

Case 13 When $U = \mathrm{sn}(\xi,k)$ and $V = \mathrm{ns}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)]\,,$$
$$\frac{d}{c} = [a^2(1+k^2) + 3c^2(1+m^2)]\,,$$
$$\alpha = \pm\sqrt{\frac{k}{m}}\,, \tag{37}$$

then the breather lattice solution is

$$\phi_{13} = \pm\sqrt{\frac{k}{m}}[\mathrm{sn}(\xi,k)\mathrm{ns}(\eta,m)]\,. \tag{38}$$

Case 14 When $U = \mathrm{cd}(\xi,k) = \mathrm{cn}(\xi,k)/\mathrm{dn}(\xi,k)$ and $V = \mathrm{ns}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)]\,,$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)]\,,$$
$$\alpha = \pm\sqrt{\frac{k}{m}}\,, \tag{39}$$

then the breather lattice solution is

$$\phi_{14} = \pm\sqrt{\frac{k}{m}}[\mathrm{cd}(\xi,k)\mathrm{ns}(\eta,m)]\,. \tag{40}$$

Case 15 When $U = \mathrm{sn}(\xi,k)$ and $V = \mathrm{dc}(\eta,m) = \mathrm{dn}(\eta,m)/\mathrm{cn}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)]\,,$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)]\,,$$
$$\alpha = \pm\sqrt{\frac{k}{m}}\,, \tag{41}$$

then the breather lattice solution is

$$\phi_{15} = \pm\sqrt{\frac{k}{m}}[\mathrm{sn}(\xi,k)\mathrm{dc}(\eta,m)]\,. \tag{42}$$

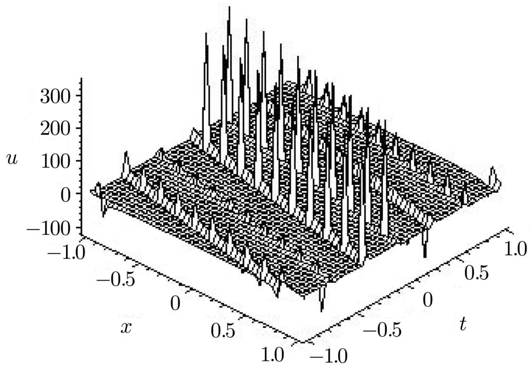

Fig. 3 The space-time evolution of the breather lattice solution of Eqs. (41) and (42). $a = 1$, $c = 1$, $m = 0.5$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = 5$, $d = 5$, $k = 0.5$, and $\alpha = 1$.

The periodic characteristics in Fig. 3 is similar to what we see in Fig. 2, but the specific direction is different. At the same time, the magnitude of u is asymmetric in the positive and the negative directions.

Case 16 When $U = \mathrm{cd}(\xi,k)$ and $V = \mathrm{dc}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}, \quad \frac{b}{a} = [a^2(1+k^2) + 3c^2(1+m^2)]\,,$$
$$\frac{d}{c} = [3a^2(1+k^2) + c^2(1+m^2)]\,,$$
$$\alpha = \pm\sqrt{\frac{k}{m}}\,, \tag{43}$$

then the breather lattice solution is

$$\phi_{16} = \pm\sqrt{\frac{k}{m}}[\mathrm{cd}(\xi,k)\mathrm{dc}(\eta,m)]\,. \tag{44}$$

Case 17 When $U = \mathrm{cn}(\xi,k)$ and $V = \mathrm{nc}(\eta,m) = 1/\mathrm{cn}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}\,,$$
$$\frac{b}{a} = -[a^2(2k^2-1) + 3c^2(2m^2-1)]\,,$$
$$\frac{d}{c} = -[3a^2(2k^2-1) + c^2(2m^2-1)]\,,$$
$$\alpha = \pm\Big[\frac{k^2(1-m^2)}{m^2(1-k^2)}\Big]^{1/4}, \tag{45}$$

then the breather lattice solution is

$$\phi_{17} = \pm\Big[\frac{k^2(1-m^2)}{m^2(1-k^2)}\Big]^{1/4}[\mathrm{cn}(\xi,k)\mathrm{nc}(\eta,m)]\,. \tag{46}$$

Case 18 When $U = \mathrm{sd}(\xi,k) = \mathrm{sn}(\xi,k)/\mathrm{dn}(\xi,k)$ and $V = \mathrm{ds}(\eta,m) = \mathrm{dn}(\eta,m)/\mathrm{sn}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\begin{aligned}\frac{a^2}{c^2} &= \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}\,,\\ \frac{b}{a} &= -[a^2(2k^2-1)+3c^2(2m^2-1)]\,,\\ \frac{d}{c} &= -[3a^2(2k^2-1)+c^2(2m^2-1)]\,,\\ \alpha &= \pm\Big[\frac{k^2(1-k^2)}{m^2(1-m^2)}\Big]^{1/4}\,,\end{aligned} \tag{47}$$

then the breather lattice solution is

$$\phi_{18} = \pm\Big[\frac{k^2(1-k^2)}{m^2(1-m^2)}\Big]^{1/4}[\mathrm{sd}(\xi,k)\mathrm{ds}(\eta,m)]\,. \tag{48}$$

Case 19 When $U = \mathrm{cn}(\xi,k)$ and $V = \mathrm{ds}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\begin{aligned}\frac{a^2}{c^2} &= \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}\,,\\ \frac{b}{a} &= -[a^2(2k^2-1)+3c^2(2m^2-1)]\,,\\ \frac{d}{c} &= -[3a^2(2k^2-1)+c^2(2m^2-1)]\,,\\ \alpha &= \pm\Big[\frac{k^2}{m^2(1-m^2)(1-k^2)}\Big]^{1/4}\,,\end{aligned} \tag{49}$$

then the breather lattice solution is

$$\phi_{19} = \pm\Big[\frac{k^2}{m^2(1-m^2)(1-k^2)}\Big]^{1/4}[\mathrm{cn}(\xi,k)\mathrm{ds}(\eta,m)]\,. \tag{50}$$

Case 20 When $U = \mathrm{nc}(\xi,k)$ and $V = \mathrm{sd}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\begin{aligned}\frac{a^2}{c^2} &= \frac{m}{k}\sqrt{\frac{1-m^2}{1-k^2}}\,,\\ \frac{b}{a} &= -[a^2(2k^2-1)+3c^2(2m^2-1)]\,,\\ \frac{d}{c} &= -[3a^2(2k^2-1)+c^2(2m^2-1)]\,,\\ \alpha &= \pm\Big[\frac{m^2(1-m^2)(1-k^2)}{k^2}\Big]^{1/4}\,,\end{aligned} \tag{51}$$

then the breather lattice solution is

$$\phi_{20} = \pm\Big[\frac{m^2(1-m^2)(1-k^2)}{k^2}\Big]^{1/4}[\mathrm{nc}(\xi,k)\mathrm{sd}(\eta,m)]\,. \tag{52}$$

Case 21 When $U = \mathrm{sn}(\xi,k)$ and $V = \mathrm{cs}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\begin{aligned}\frac{a^2}{c^2} &= \sqrt{\frac{1-m^2}{k^2}}\,,\\ \frac{b}{a} &= [a^2(1+k^2)-3c^2(2-m^2)]\,,\\ \frac{d}{c} &= [3a^2(1+k^2)-c^2(2-m^2)]\,,\\ \alpha &= \pm\Big[\frac{k^2}{1-m^2}\Big]^{1/4}\,,\end{aligned} \tag{53}$$

then the breather lattice solution is

$$\phi_{21} = \pm\Big[\frac{k^2}{1-m^2}\Big]^{1/4}[\mathrm{sn}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{54}$$

When $k \to 1$ and $m \to 0$, the breather lattice solution (54) turns to be a solution

$$\phi_{21'} = \pm[\tanh(\xi)\cot(\eta)]\,, \tag{55}$$

with

$$a^2 = c^2, \quad b = -\frac{\gamma}{4a}, \quad d = \pm\frac{\gamma}{4a}\,. \tag{56}$$

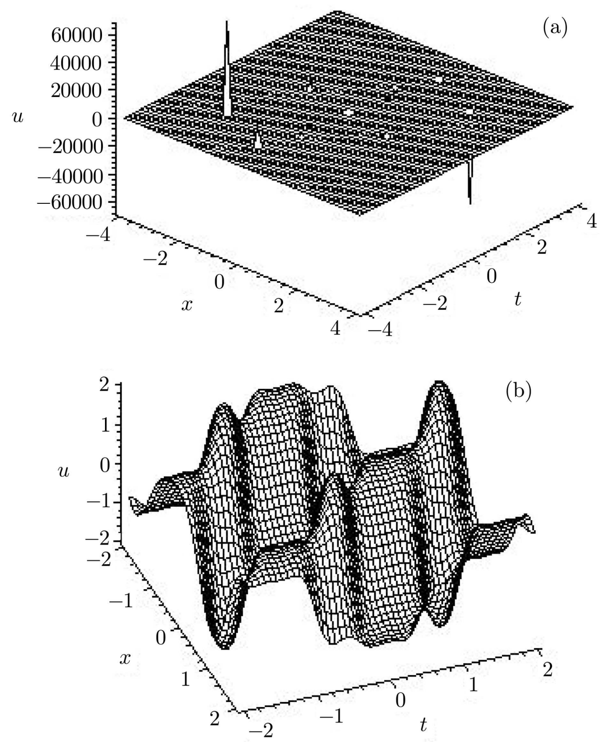

Fig. 4 The space-time evolution of the breather lattice solution of Eqs. (53) and (54). (a) $a = 1$, $c = 1$, $m = 0.8$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = -2.72$, $d = 2.72$, $k = 0.6$, and $\alpha = 1$; (b) $a = 1$, $c = 1$, $m = 0$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = -4$, $d = 4$, $k = 1$, and $\alpha = 1$.

From Fig. 4, it is obvious that for different values of m and k, the same breather lattice solution will also show different characteristics, small or large. Especially, when m and k take their limiting values, the behavior will be quite different from what given in Fig. 4. Actually, the characteristics of the breather lattice solution shown in the upper panel of Fig. 4 is sporadic, the magnitude of u has an antisymmetric characteristics along a specific direction. But for the bottom panel of Fig. 4, the profiles in any directions are smooth.

Case 22 When $U = \mathrm{cd}(\xi,k)$ and $V = \mathrm{cs}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[\frac{k^2}{1-m^2}\Big]^{1/4}, \tag{57}$$

then the breather lattice solution is

$$\phi_{22} = \pm\Big[\frac{k^2}{1-m^2}\Big]^{1/4}[\mathrm{cd}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{58}$$

Case 23 When $U = \mathrm{ns}(\xi,k)$ and $V = \mathrm{cs}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[\frac{1}{k^2(1-m^2)}\Big]^{1/4}, \tag{59}$$

then the breather lattice solution is

$$\phi_{23} = \pm\Big[\frac{1}{k^2(1-m^2)}\Big]^{1/4}[\mathrm{ns}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{60}$$

Case 24 When $U = \mathrm{dc}(\xi,k)$ and $V = \mathrm{cs}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[\frac{1}{k^2(1-m^2)}\Big]^{1/4}, \tag{61}$$

then the breather lattice solution is

$$\phi_{24} = \pm\Big[\frac{1}{k^2(1-m^2)}\Big]^{1/4}[\mathrm{dc}(\xi,k)\mathrm{cs}(\eta,m)]\,. \tag{62}$$

Case 25 When $U = \mathrm{dc}(\xi,k)$ and $V = \mathrm{sc}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[\frac{1-m^2}{k^2}\Big]^{1/4}, \tag{63}$$

then the breather lattice solution is

$$\phi_{25} = \pm\Big[\frac{1-m^2}{k^2}\Big]^{1/4}[\mathrm{dc}(\xi,k)\mathrm{sc}(\eta,m)]\,. \tag{64}$$

Case 26 When $U = \mathrm{ns}(\xi,k)$ and $V = \mathrm{sc}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[\frac{1-m^2}{k^2}\Big]^{1/4}, \tag{65}$$

then the breather lattice solution is

$$\phi_{26} = \pm\Big[\frac{1-m^2}{k^2}\Big]^{1/4}[\mathrm{ns}(\xi,k)\mathrm{sc}(\eta,m)]\,. \tag{66}$$

Case 27 When $U = \mathrm{sn}(\xi,k)$ and $V = \mathrm{sc}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,\quad \alpha = \pm\Big[k^2(1-m^2)\Big]^{1/4}, \tag{67}$$

then the breather lattice solution is

$$\phi_{27} = \pm[k^2(1-m^2)]^{1/4}[\mathrm{sn}(\xi,k)\mathrm{sc}(\eta,m)]\,. \tag{68}$$

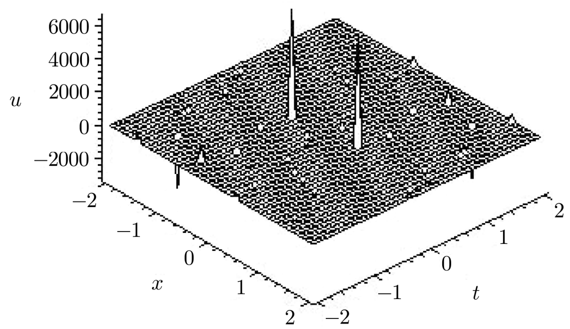

Fig. 5 The space-time evolution of the breather lattice solution of Eqs. (69) and (70). $a = 1$, $c = 1$, $m = 0.6$, $\xi_0 = \eta_0 = 0$, from which the other parameters can be determined as $b = -3.28$, $d = 3.28$, $k = 0.8$, and $\alpha = 0.8$.

Case 28 When $U = \mathrm{cd}(\xi,k)$ and $V = \mathrm{sc}(\eta,m)$, then from (7) and (9), the parameters can be determined as

$$\frac{a^2}{c^2} = \sqrt{\frac{1-m^2}{k^2}}\,,\quad \frac{b}{a} = [a^2(1+k^2) - 3c^2(2-m^2)]\,,\quad \frac{d}{c} = [3a^2(1+k^2) - c^2(2-m^2)]\,,$$

$$\alpha = \pm[k^2(1-m^2)]^{1/4}\,, \qquad (69)$$

then the breather lattice solution is

$$\phi_{28} = \pm[k^2(1-m^2)]^{1/4}[\mathrm{cd}(\xi,k)\mathrm{sc}(\eta,m)]\,. \qquad (70)$$

Similar to what seen in the upper panel of Fig. 4, Fig. 5 takes a sporadic behavior, only within some specific regions, the magnitude of u is visible.

3 Conclusion

In this paper, dependent and independent variable transformations are introduced to solve the negative mKdV equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that besides the solutions expressed in terms of the different combinations of Jacobi elliptic functions, there are solutions expressed in terms of elementary functions, which can be obtained in the above solutions in the limit cases where k and/or m take the value 0 and/or 1. However, not all the combinations of Jacobi elliptic functions are the solutions to the negative mKdV equation (3), only these ones that can satisfy the constraints (10) can be the solutions to the negative mKdV equation (1). Furthermore, when different independent variable transformations are adopted, there will be different results. For example, when we choose the independent variable transformation

$$\xi = ax + \frac{1}{a}t + \xi_0, \quad \eta = ax - \frac{1}{a}t + \eta_0\,, \qquad (71)$$

which is given in Ref. [2], some breather lattice solutions expressed in terms of Jacobi elliptic functions will be omitted. Under variable transformations mentioned above, all solutions can be expressed in terms of 12 basic Jacobi elliptic functions are listed in this paper, there are only 28 combinations of Jacobi elliptic functions can satisfy the constraints (10). Due to wide applications of the negative mKdV equation, the analytical solutions given in this paper will be helpful in related research.

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