角动量理论

*) 1925年Heisenberg和Jordan使用代数解法求解
但在1913年数学家Eric Cartan已经给出了答案。

*) 角动量对易关系

$$[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k$$

- $\hbar$没有任何用处，仅仅提供一个量纲而已
  定义 $J_i' = J_i/\hbar \Longrightarrow [J_i', J_j'] = iJ_k'$

- $[\hat{J}^2, \hat{J}_z] = 0$

  J^2并不依赖于系统的轴向，因为体系的三维旋转不变性
  使得我们无法区分$J_{x,y,z}$分量。

- 定义 $J_\pm = J_x \pm iJ_y$，升降算符

$$J_+ = (J_x - iJ_y)^+ = (J_-)^+$$

$$[J_\pm, J_z] = [J_x \pm iJ_y, J_z] = [J_x, J_z] \mp i[J_y, J_z]$$

$$= -i\hbar J_y \mp i(i\hbar)J_x = -i\hbar J_y \pm \hbar J_x$$

$$= \pm\hbar[J_x \mp iJ_y] = \pm\hbar J_\mp$$

$$[J_-, J_+] = [J_x - iJ_y, J_x + iJ_y] = -i[J_y, J_x] + i[J_x, J_y]$$

$$= 2i[J_x, J_y] = 2i(i\hbar)J_z = -2\hbar J_z$$

$$[J^2, J_\pm] = 0$$

*) $[\hat{J}^2, \hat{J}_i] = 0$ 其中 $J^2 = J_x^2 + J_y^2 + J_z^2$

$$[J^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z] = [J_x^2 + J_y^2, J_z]$$

$$[J_y^2, J_z] = J_y[J_y, J_z] + [J_y, J_z]J_y = J_y(i\hbar J_x) + (i\hbar)J_x J_y = i\hbar(J_y J_x + J_x J_y)$$

$$[J_x^2, J_z] = J_x[J_x, J_z] + [J_x, J_z]J_x = -i\hbar(J_x J_y + J_y J_x)$$

$$\Rightarrow [J_x^2 + J_y^2, J_z] = 0$$

$$\Rightarrow [J^2, J_z] = 0$$ 同理可得 $[J^2, J_x] = [J^2, J_y^2] = 0$

$\hat{J}^2$ 和 $\hat{J}_z$ 具有共同本征函数

*) J^2和J_z共同本征函数

① $[L]$=能量×时间=$[\hbar]$，所以我们定义

$$\hat{J}^2|jm\rangle = \eta_j \hbar^2 |jm\rangle$$

$$\hat{J}_z|jm\rangle = m\hbar|jm\rangle$$

其中$\underline{\eta_j}$和$\underline{m}$为无量纲数

所以

$$(J^2-J_z^2)|jm\rangle = (\eta_j - m^2)\hbar^2|jm\rangle$$

因为 $J^2-J_z^2 = J_x^2+J_y^2$，且 $\overline{J_x^2+J_y^2} \geq 0$

$$\Rightarrow \eta_j \geq m^2 \text{ 或 } |m| \leq \sqrt{\eta_\ell}$$

② 因为 $[J_z, J_-] = -\hbar J_-$，所以

$$J_z \underline{J_-|jm\rangle} = (J_- J_z - \hbar J_-)|jm\rangle = J_-(J_z - \hbar)|jm\rangle = (m-1)\hbar \underline{J_-|jm\rangle}$$

$\Rightarrow \hat{J}_-|jm\rangle$ 也是 J^2 和 J_z 的本征函数，本征值为 $\eta_j \hbar^2$ 和 $(m-1)\hbar$

$\Rightarrow J_-$ 为降标符

③ 同理：

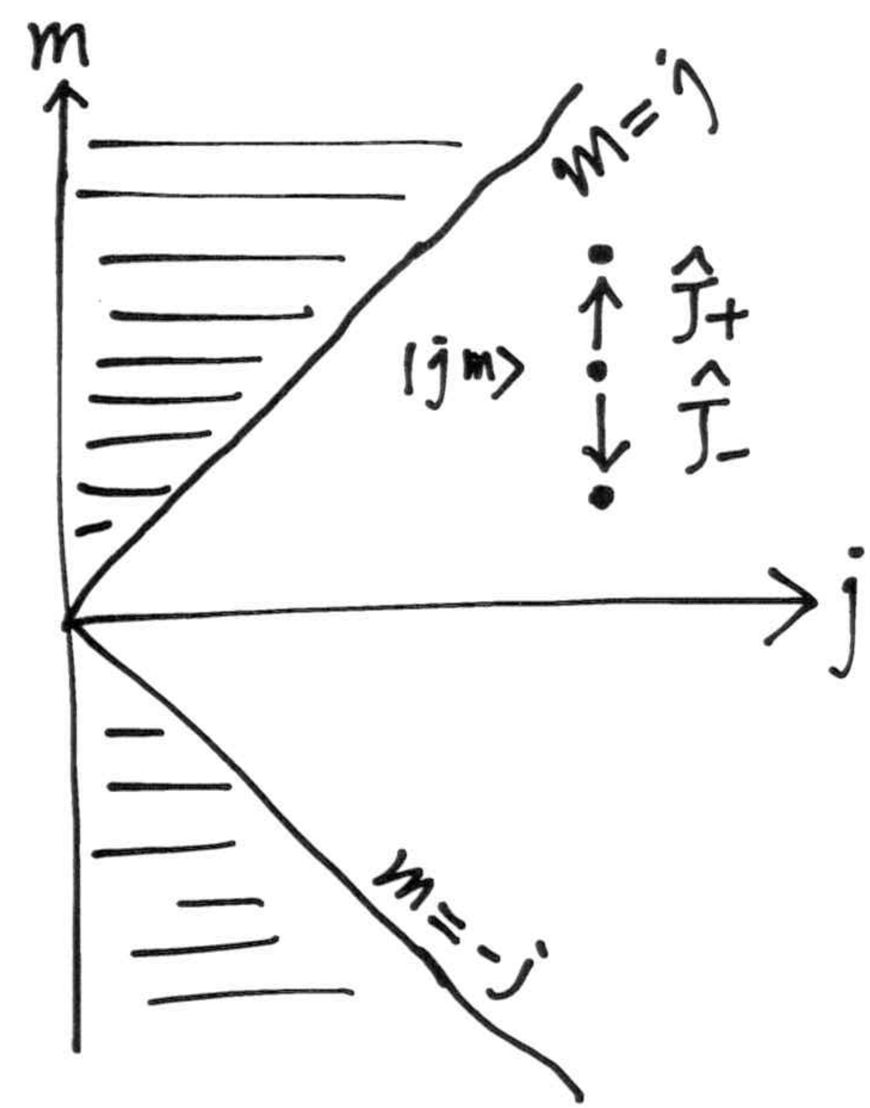
$$\hat{J}_z \hat{J}_+ |jm\rangle = \hat{J}_+ (\hat{J}_z + \hbar)|jm\rangle = (m+1)\hbar \hat{J}_+|jm\rangle$$

$\Rightarrow \hat{J}_+$ 为升标符

*) 升降算符图示

从上面讨论可知，由任意一个 $\hat{J}^2$ 和 $\hat{J}_z$ 的本征矢量 $|jm\rangle$ 出发，重复使用 $\hat{J}_+$ 或 $\hat{J}_-$ 可得到 $\hat{J}^2$ 算符的相同本征值 j 的一系列矢量，这组矢量都属于 $\hat{J}_z$ 的本征矢，但本征值 m 相差整数单位。

考虑 $\hat{J}_\pm|jm\rangle$ 的 Norm



$$\hat{J}_\mp \hat{J}_\pm = (\hat{J}_x \mp i\hat{J}_y)(\hat{J}_x \pm i\hat{J}_y)$$

$$= \hat{J}_x^2 + \hat{J}_y^2 \pm i[\hat{J}_x, \hat{J}_y]$$

$$= \hat{J}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z$$

$$\Rightarrow \|\hat{J}_\pm |jm\rangle\|^2 = \langle jm|\hat{J}_\mp \hat{J}_\pm|jm\rangle$$

$$= [j(j+1) - m(m\pm1)]\hbar^2$$

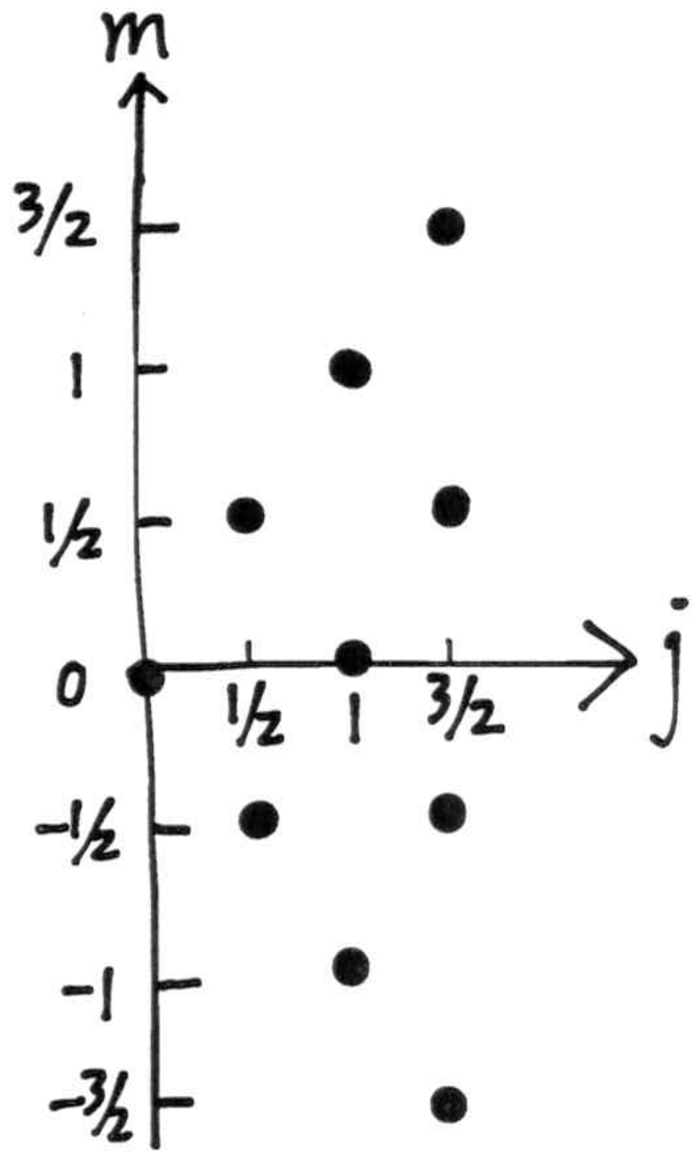
选取合适的 $|jm\rangle$ 相位使得

$$\hat{J}_{\pm}|jm\rangle = \sqrt{j(j+1)-m(m\pm1)}\,\hbar\,|j, m\pm1\rangle$$

当 $m=j$ 或 $-j$ 时,

$$\hat{J}_{+}|j, j\rangle = 0$$

$$\hat{J}_{-}|j, -j\rangle = 0$$



④ 设 m 的最大值为 m_+，最小值为 m_-，即

$$\hat{J}_+|jm\rangle = 0, \quad \hat{J}_-|jm\rangle = 0$$

所以 $$\hat{J}_-\hat{J}_+|jm_+\rangle = 0, \quad \hat{J}_+\hat{J}_-|jm_-\rangle = 0$$

$$\Rightarrow (\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|jm_+\rangle = (\eta_j - m_+^2 - m_+)\hbar^2|jm_+\rangle = 0$$

$$(\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z)|jm_-\rangle = (\eta_j - m_-^2 + m_-)\hbar^2|jm_-\rangle = 0$$

$$\Rightarrow \eta_j = m_+(m_+ + 1) = m_-(m_- - 1)$$

$$m_+^2 + m_+ = m_-^2 - m_- \Rightarrow (m_+ + m_-)(m_+ - m_-) + (m_+ + m_-) = 0$$

$$\Rightarrow (m_+ + m_-)(m_+ - m_- + 1) = 0 \Rightarrow \underline{m_+ = -m_-} \text{ 或者 } \underline{m_+ = m_- - 1}$$

×

记 $m_+ \equiv j$，则有 $\eta_j = j(j+1)$

⑤ 因为 $\hat{J}_\pm$ 算符使我们可以走遍 $\hat{J}_z$ 算符的完整希尔伯特空间

所以将 $\hat{J}_z$ 算符作用 N 次后我们可以从 $m_+ \to m_-$，此即

$$j - N = -j \Longrightarrow j = \frac{N}{2} \quad \text{(其中 N 是整数)}$$

此时 $\hat{J}_z$ 算符张开的希尔伯特空间的维数为 $2j+1 = N+1$

$$m \in \{-j, -j+1, \cdots, j-1, j\}$$

（j 为整数或半整数）

轨道角动量(L)

波函数的周期性边界条件要求 m 为整数

$$m = 0, \pm 1, \pm 2, \cdots$$

*) $l = 0$ 态:

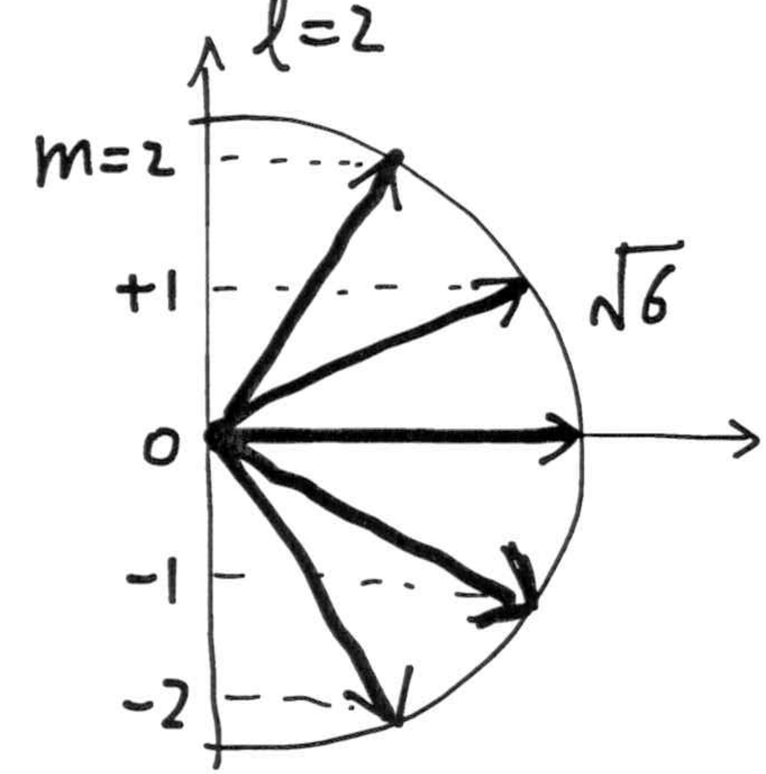
经典物理中 $\vec{l} = \vec{r} \times \vec{p} \Longrightarrow \vec{r} \parallel \vec{p}$ (此即为粒子沿一条通过原点的直线振荡)

量子物理中,"轨道"的概念被摈弃,但有些术语名词,例如轨道角动量,轨道磁矩等名词仍然继续使用,只不过是为了方便而已。

*) 经典物理认为：一个矢量在某方向上最大的投影就是它的大小，该矢量平方为l^2；

而量子理论告诉我们：角动量l在某方向投影为$m=l$，但它的平方却是$l(l+1)$。

Feynman将之解释为空间量子化：



$\hat{J}_z$的$(2l+1)$维本征矢量构成$\hat{J}^2$的子空间，

m可取2l+1个离散值 $-l, -l+1, \cdots, l-1, l$

从而$\hat{l}_z^2$的平均值为

$$\overline{l_z^2} = \frac{1}{2l+1}\sum_{m=-l}^{l} m^2 = \frac{1}{3}l(l+1)$$

区 $\left(\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)\right)$

由各向同性而知

$$\overline{l^2} = 3\overline{l_z^2} = l(l+1)$$

⇒ 如图所示，球半径为$\sqrt{l(l+1)}$，其在z方向的整数投影最大值仅为$\pm l$ （$l=0$情况例外）

*) 问：为何我们不能选取角动量矢量方向作z轴呢？

这时应该有 $m_+ = \sqrt{l(l+1)}$

答：因为一个自由粒子根本没有一个确定的角动量矢量，因为 $\hat{L}_x, \hat{L}_y, \hat{L}_z$ 不对易，所以它们无法同时确定，就象自由粒子没有同时确定的坐标和动量一样。

如果角动量分量完全沿着z方向，则 $\hat{L}_x = \hat{L}_y = 0$，$\hat{L}_z = \sqrt{l(l+1)}\hbar$

$\Rightarrow$ $\hat{L}_x, \hat{L}_y, \hat{L}_z$ 可同时确定，违背不确定关系。

注意：三维时空中旋转操作是非阿贝尔的 (non-abelian)

本征值 l_x, l_y, l_z 中只要有1个不等于0，$\hat{L}_x, \hat{L}_y, \hat{L}_z$ 就没有共同本征函数

*) 量子"涨落"

在 $\hat{L}_z$ 本征态中，$\hat{L}_x$ 和 $\hat{L}_y$ 不确定，其涨落为

$$\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \frac{1}{2}\langle \hat{L}^2 - L_z^2 \rangle = \frac{1}{2}\left[l(l+1) - m^2 \right]\hbar^2 \geqslant 0$$

（其中等号仅在 $l=m=0$ 时成立）

$$\Delta L_x = \sqrt{\langle L_x^2 \rangle}, \quad \Delta L_y = \sqrt{\langle L_y^2 \rangle}, \quad \langle L_x \rangle = \langle L_y \rangle = 0$$

- 当体系处于 $\hat{L}_z$ 本征值态时，

$$\Delta L_x \cdot \Delta L_y = \sqrt{\langle L_x^2 \rangle}\sqrt{\langle L_y^2 \rangle} \geqslant \frac{\hbar}{2}\left|\langle L_z \rangle\right| = \frac{1}{2} m\hbar^2$$

$$\Rightarrow \Delta L_x \cdot \Delta L_y = \frac{1}{2}\left[l(l+1) - m^2 \right]\hbar^2 \geqslant \frac{1}{2} m\hbar^2$$

（因为 $m \leq l$，所以上式成立；当 $m=l$ 时，等式成立）

- 此时 $\Delta\hat{L}_z=0$，但 $\hat{L}_x$ 和 $\hat{L}_y$ 的不确定度为有限

- 在 l_z 的 $m=l$ 本征态中，$\langle L_x^2\rangle=\langle L_y^2\rangle=\frac{1}{2}\hbar^2$

因为 $L_{x,y}^2$ 与体系能量相关，所以不确定关系告诉我们

在 $\hat{L}_z$ 本征态中，虽然 $\langle L_x\rangle=\langle L_y\rangle=0$，

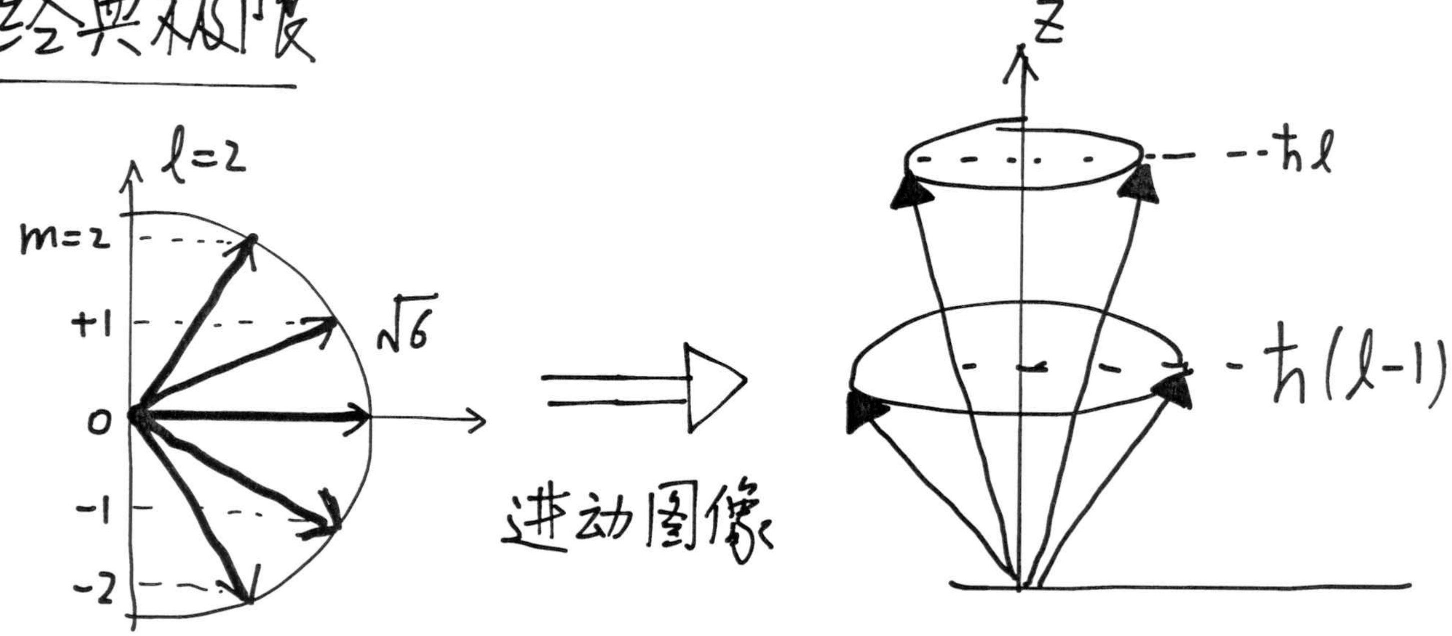
但在 x-y 方向上仍有不为 0 的能量。

- 角动量是全空间的概念，也告诉我们物理体系在空间中几率分布的对称性质。非束缚粒子将充满整个空间。

- 角动量是 3 维空间中才具有的。

当$\hbar\to 0$时，不确定关系将不再限制任何物理可观测量。

*) 经典极限



回到经典物理的具体操作是

(1) $l\to\infty$（宏观尺度下物理自由度无限多）

$\Rightarrow l(l+1)\approx l^2=(l_z^2)_{max}$（此为经典矢量图像）

(2) $\hbar\to 0$ 但保证 $l\hbar$ 有限

但自旋角动量S无法取为∞，所以当$\hbar\to 0$时自旋无经典对应。

$$S^2=s(s+1)=\frac{3}{4}\ ,\ \ S_z=\pm\frac{1}{2}$$

轨道角动量

因为角动量$\vec{L}$在体系膨胀时不变，所以它仅仅作用在角度上。
在19世纪角动量的本征函数和本征值由Legendre和Fourier给出。

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

其共同本征函数为球谐函数(spherical harmonics) $Y_l^m(\theta,\varphi)$

$$\hat{L}^2 Y_l^m(\theta,\varphi) = l(l+1)\hbar^2 Y_l^m(\theta,\varphi)$$

$$\hat{L}_z Y_l^m(\theta,\varphi) = m\hbar Y_l^m(\theta,\varphi)$$

其中 $Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\,\underline{P_l^m(\cos\theta)}e^{im\varphi}$

$$P_l^m(\cos\theta) = (-1)^{l+m}\frac{1}{2^l l!}\frac{(l+m)!}{(l-m)!}\frac{1}{\sin^m\theta}\left(\frac{d}{d\cos\theta}\right)^{l-m}\sin^{2l}\theta$$

(associated Legendre function)

- 平方可积的球谐函数形成了一个希尔伯特空间 (radius = 1)

① 正交归一

$$\iint \left(Y_l^m(\theta,\varphi)\right)^* Y_{l'}^{m'}(\theta,\varphi)\sin\theta \, d\theta \, d\varphi = \delta_{ll'}\delta_{mm'}$$

② 封闭性

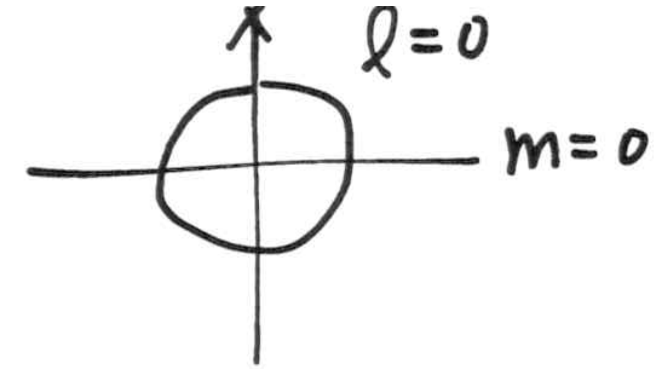
$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\varphi)\left(Y_l^m(\theta',\varphi')\right)^* = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\varphi-\varphi')$$

③ 递推关系

$$L_\pm Y_l^m = \sqrt{l(l+1)-m(m\pm 1)}\,\hbar Y_l^{m+1}$$

$$= \sqrt{(l\mp m)(l\pm m+1)}\,\hbar Y_l^{m+1}$$

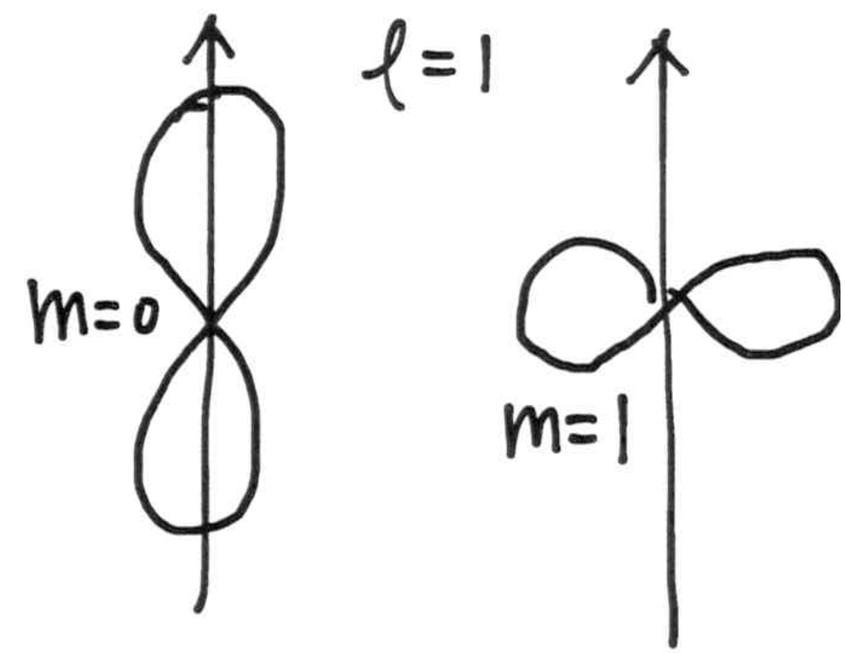
④ $l=0$: $Y_0^0(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$



$l=1$: $Y_1^1(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$

$Y_1^0(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$

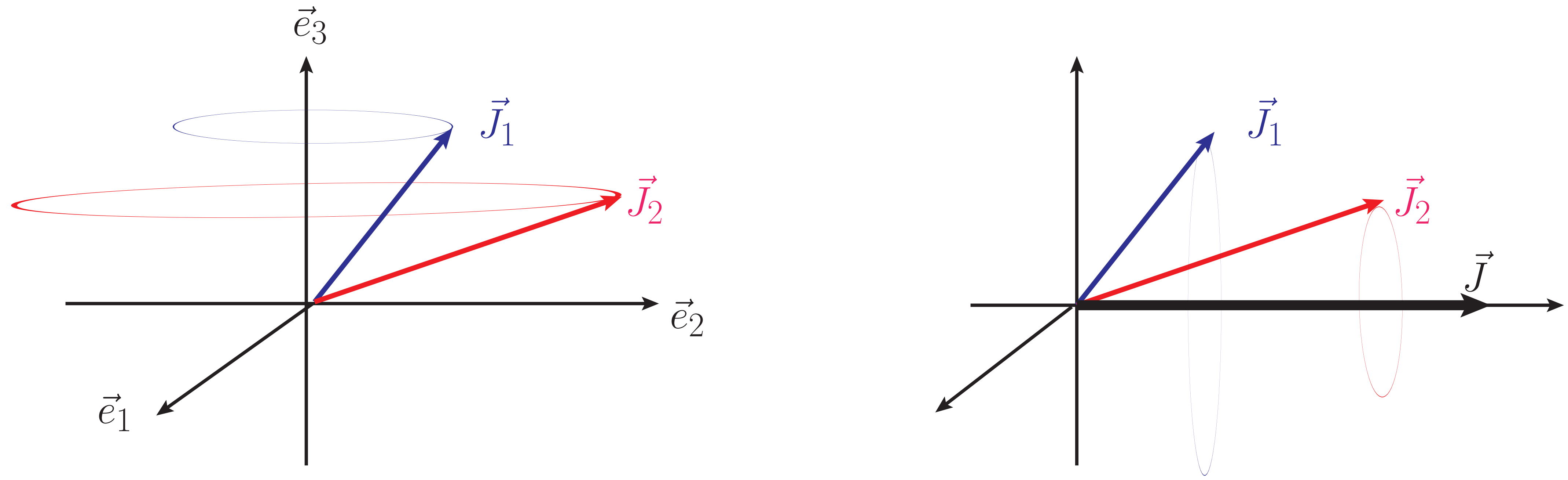
$Y_1^{-1}(\theta,\varphi)=\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$



容易验证:

$\sum_{m=-1}^{1}|Y_1^m(\theta,\varphi)|^2$ 不依赖于θ和φ角，各向同性

角动量耦合和
Clebsch-Gordon系数

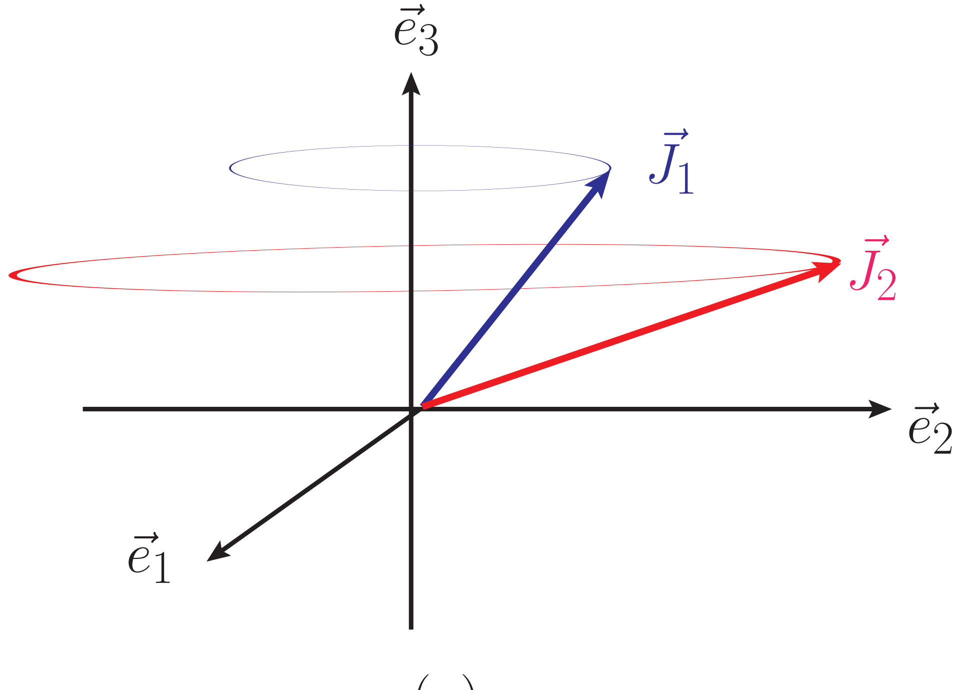


(a) (b)

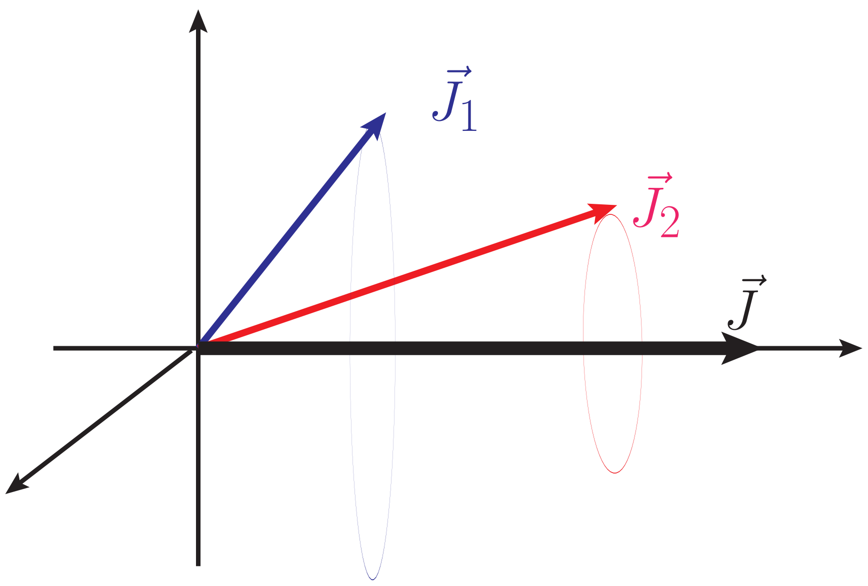
经典物理中，两个物体的角动量是作用在同一空间中，因此总角动量等于各自角动量分量之和

量子力学中，两个物体的角动量作用在不同的希尔伯特空间

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$$



(a)



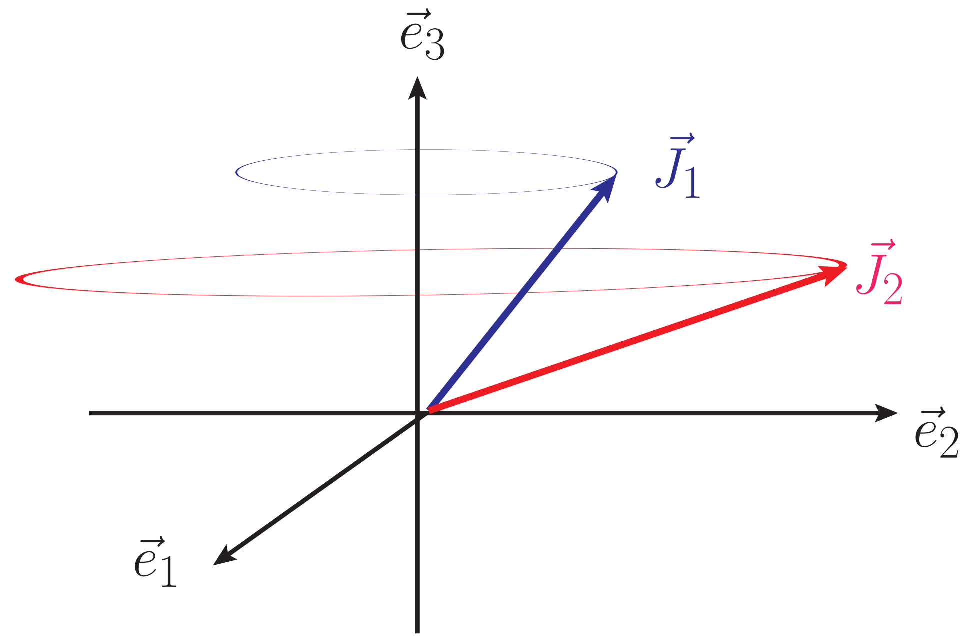
(b)

$$\vec{J} = \vec{J}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \vec{J}_2 \equiv \vec{J}_1 + \vec{J}_2$$

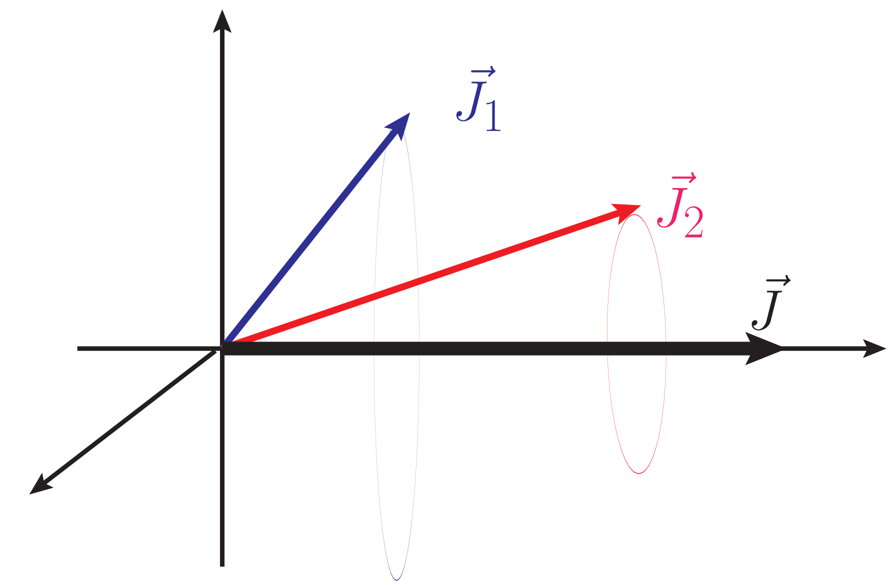
$$[\vec{J}_1, \vec{J}_2] = 0$$

$$\begin{aligned} [J_i, J_j] &= [J_{1i} + J_{2i}, J_{1j} + J_{2j}] = [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] \\ &= i\hbar\epsilon_{ijk} J_{1k} + i\hbar\epsilon_{ijk} J_{2k} = i\hbar\epsilon_{ijk} \left(J_{1k} + J_{2k}\right) \\ &= i\hbar\epsilon_{ijk} J_k \end{aligned}$$

$$[\vec{J}^2, J_z] = 0$$



(a)



(b)

$$\{\vec{J}_1^2, \hat{J}_{1z}, \vec{J}_2^2, \vec{J}_{2z}\}$$

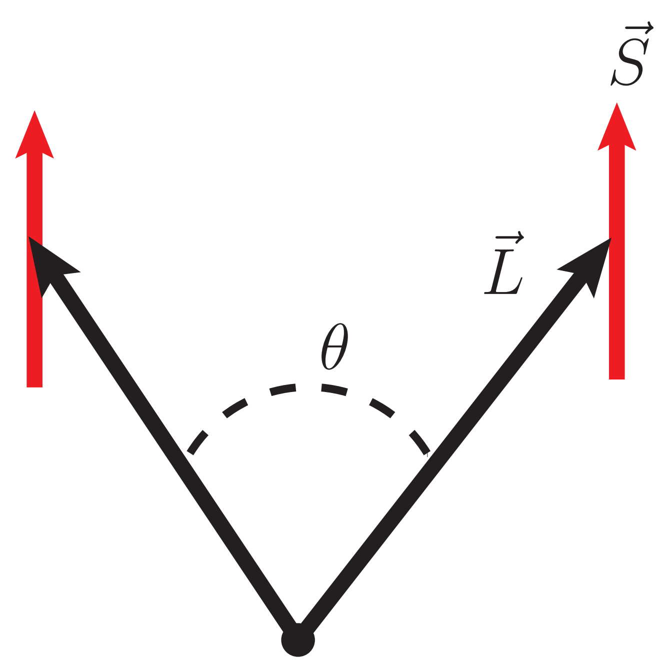
因子化基矢

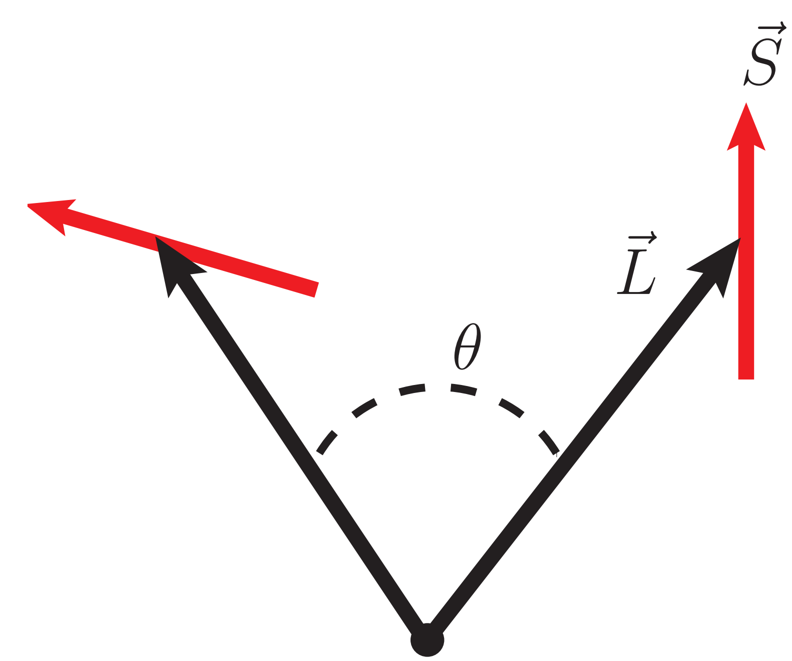
$$\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1 m_1; j_2 m_2\rangle\}$$

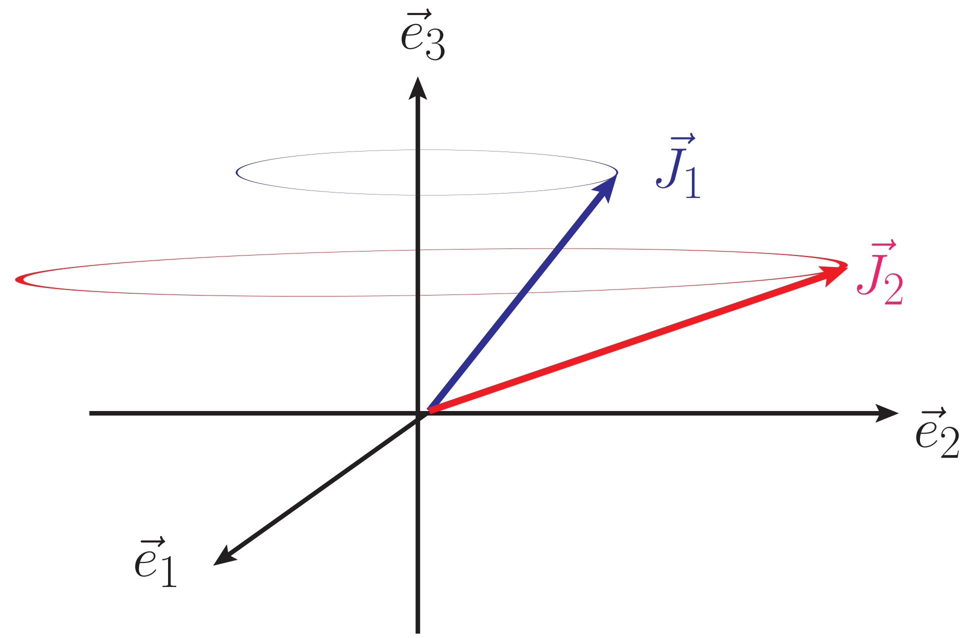
$$\{\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, J_z\}$$

耦合基矢

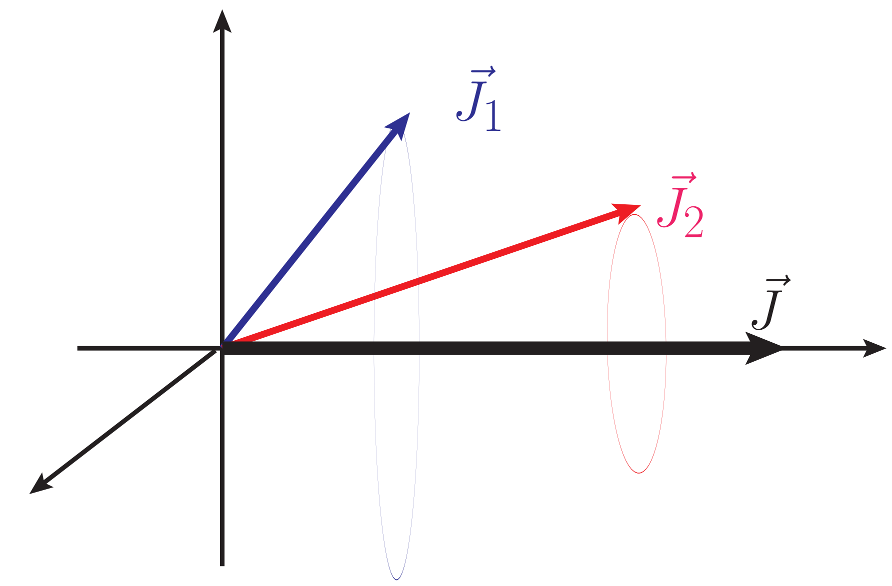
$$\{|j_1, j_2, j, m_j\rangle\}$$







(a)



(b)

总角动量为每个子系统角动量的量子化提供参考方向

$$\vec{J}_1 = \vec{J} - \vec{J}_2$$

$$\vec{J}_1^2 = (\vec{J} - \vec{J}_2)^2 = \vec{J}^2 + \vec{J}_2^2 - 2\vec{J}_2 \cdot \vec{J}$$

$$J_1(J_1+1) = J(J+1) + J_2(J_2+1) - \frac{2\vec{J}_2 \cdot \vec{J}}{\hbar^2}$$

$$\vec{J}_2 \cdot \vec{J} = \frac{J(J+1) + J_2(J_2+1) - J_1(J_1+1)}{2}\hbar^2$$

耦合基矢和因子化基矢可以完全描述相同的希尔伯特空间，所以两者是等价的。

维度：$(2j_1+1)(2j_2+1)$

下面我们讨论在因子化基矢张开的子空间中 $\vec{J}^2$ 和 $\hat{J}_z$ 的本征值和本征矢量形式，或者说，讨论两种基矢之间的转化关系。

$$|j_1,j_2,j,m_j\rangle = \sum_{m_1,m_2} C^{j,m}_{j_1,m_1,j_2,m_2}\,|j_1,m_1;j_2,m_2\rangle$$

Clebsch-Gordon系数：两套基矢之间的变换矩阵

$$C^{jm}_{j_1m_1j_2m_2} = \langle j_1m_1;j_2,m_2\,|j_1j_2jm_j\rangle$$

C-G系数

因为耦合基矢是 J_z 的本征态，而且 $J_z \leq J$，所以角动量耦合的总角动量的最大值应该是 J_{1z} 和 J_{2z} 的最大值之和。总角动量在 z 方向分量最大的态和因子化基矢之间具有如下关系：

$$\left|J^{\mathrm{Max}}, J_z^{\mathrm{Max}}\right\rangle \equiv |j, j\rangle = |j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle. \tag{8.5.8}$$

$$\begin{aligned} J_{\pm} &\equiv J_x \pm iJ_y = (J_{1x} + J_{2x}) \pm i(J_{1y} + J_{2y}) \\ &= (J_{1x} \pm iJ_{1y}) \otimes I_2 + I_1 \otimes (J_{2x} \pm iJ_{2y}) \\ &= J_{1-} \otimes I_2 + I_1 \otimes J_{2-} \end{aligned}$$

$$J_{-} |j, j\rangle = (J_{1-} + J_{2-}) |j_1, j_1\rangle \otimes |j_2, j_2\rangle$$

利用 $J_{\pm}|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle$

和 $J_{-}|j,j\rangle = (J_{1-}+J_{2-})\,|j_1,j_1\rangle\otimes|j_2,j_2\rangle$

$$J_{-}|j,j\rangle = \sqrt{2j}\,|j,j-1\rangle$$

$$J_{1-}|j_1,j_2\rangle\otimes|j_2,j_2\rangle + |j_1,j_1\rangle\otimes J_{2-}|j_2,j_2\rangle$$

$$= \sqrt{2j_1}\,|j_1,j_1-1\rangle|j_2,j_2\rangle + \sqrt{2j_2}\,|j_1,j_1\rangle|j_2,j_2-1\rangle$$

$$|j,j-1\rangle = \sqrt{\frac{j_1}{j}}\,|j_1,j_1-1\rangle|j_2,j_2\rangle + \sqrt{\frac{j_2}{j}}\,|j_1,j_1\rangle|j_2,j_2-1\rangle$$

CG系数

$$|j,j\rangle \xrightarrow{J_-} |j,j-1\rangle \xrightarrow{J_-} |j,j-2\rangle\cdots|j,-j+1\rangle \xrightarrow{J_-} |j,-j\rangle$$

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \qquad (2j_1+2j_2+1)$$

下一个子空间的最高态是 $|j-1,j-1\rangle$。因为

$$\hat{J}^2|j-1,j-1\rangle = j(j-1)\hbar^2|j-1,j-1\rangle, \tag{8.5.17}$$

即，$|j-1,j-1\rangle$ 和 $|j,j-1\rangle$ 属于 $\hat{J}^2$ 算符的不同本征值的本征矢，所以 $|j-1,j-1\rangle$ 和 $|j,j-1\rangle$ 正交。从此可得

$$|j-1,j-1\rangle = \sqrt{\frac{j_2}{j_1}}|j_1,j_1-1\rangle|j_2,j_2\rangle - \sqrt{\frac{j_1}{j}}|j_1,j_1\rangle|j_2,j_2-1\rangle. \tag{8.5.18}$$

可验证

$$\langle j,j-1\,|j-1,j-1\rangle = \sqrt{\frac{j_1 j_2}{j^2}} - \sqrt{\frac{j_1 j_2}{j^2}} = 0. \tag{8.5.19}$$

在利用角动量降算符 $\hat{J}_-$，

$$|j-1,j-1\rangle \xrightarrow{J_-} |j-1,j-2\rangle \xrightarrow{J_-} |j-1,j-3\rangle \cdots |j-1,-j\rangle \xrightarrow{J_-} \big|j-1,-(j-1)\big\rangle. \tag{8.5.20}$$

下面我们验证希尔伯特的独立基矢个数。因子化基矢的独立基矢个数是 $(2j_1+1)(2j_2+1)$，所以耦合基矢的对立基矢个数也一定是 $(2j_1+1)(2j_2+1)$。在耦合基矢中，设 $j_1 > j_2$，我们发现总角动量取值和其独立基矢个数是

$$\text{总计 } 2j_2+1 \text{ 行} \quad \begin{array}{ll} \hat{J}\text{本征值} & \text{独立基矢个数} \\ j_1+j_2, & 2(j_1+j_2)+1 \\ j_1+j_2-1, & 2(j_1+j_2)+1-2 \\ j_1+j_2-2, & 2(j_1+j_2)+1-2\times 2 \\ \vdots & \vdots \\ j_1-j_2, & 2(j_1+j_2)+1-2\times(2j_2) \end{array} \tag{8.5.21}$$

$\hat{J}$本征值	独立基矢个数	颠倒独立基矢个数
$j_1+j_2,$	$2(j_1+j_2)+1$	$2(j_1+j_2)+1-4j_2$
$j_1+j_2-1,$	$2(j_1+j_2)+1-2$	$2(j_1+j_2)+1-4j_2+2$
$j_1+j_2-2,$	$2(j_1+j_2)+1-2\times 2$	$2(j_1+j_2)+1-4j_2+4$
$\vdots$	$\vdots$	$\vdots$
$j_1-j_2,$	$2(j_1+j_2)+1-2\times(2j_2)$	$2(j_1+j_2)+1$

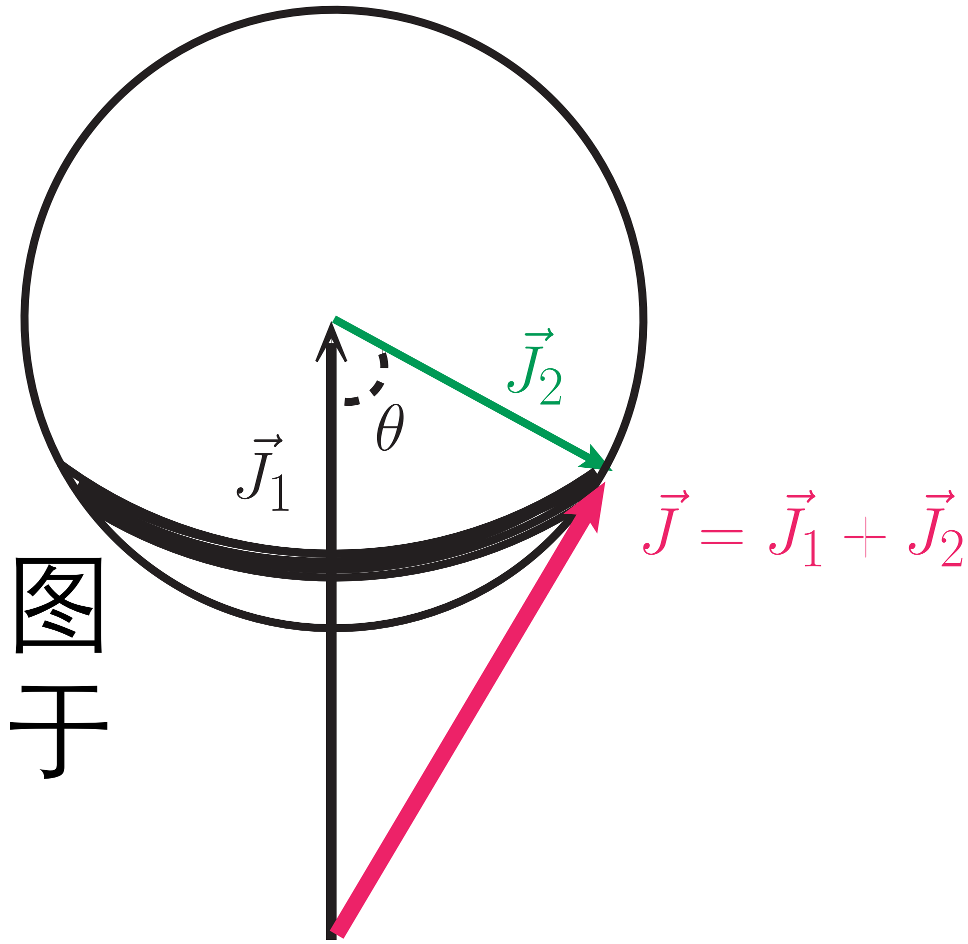
独立基矢个数是

$$\frac{1}{2}(4j_1+2)(2j_2+1)=(2j_1+1)(2j_2+1)$$

$$f=\frac{2j'+1}{(2j_1+1)(2j_2+1)}\xrightarrow[j'\gg 1 \quad j_{1,2}\gg 1]{\text{经典极限}}\frac{j'}{2j_1j_2}$$

角动量耦合的经典极限

$$\vec{J}^2 = \left(\vec{J}_1 + \vec{J}_2\right)^2 = J_1^2 + J_2^2 + 2J_1J_2\cos\theta$$



总角动量出现在 $\theta - \theta + d\theta$ 范围（图中所示黑色环形带）内的几率正比于此环形带的面积，

$$dP(\theta) = A2\pi(J_2\sin\theta)d\theta$$

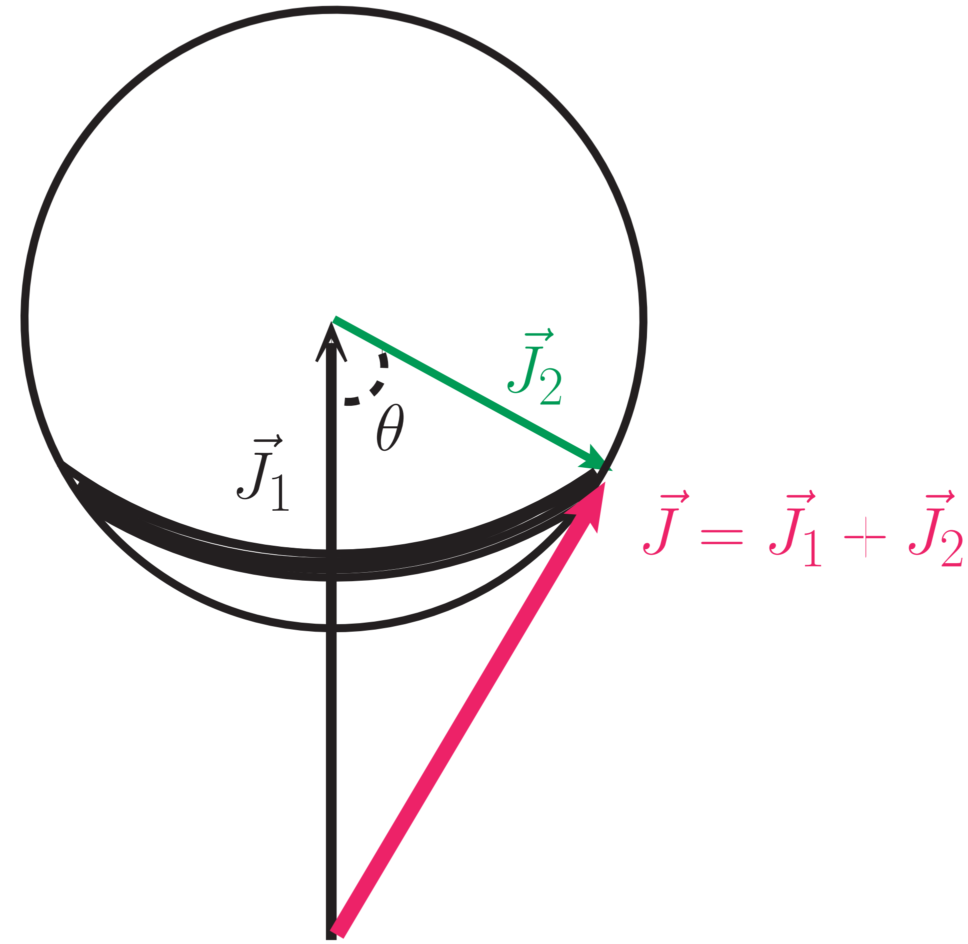
$$1 = \int dP(\theta) = A2\pi J_2(-\cos\theta)\Big|_0^{-1} = A4\pi J_2$$

归一化的几率密度为 $dP(\theta) = \frac{1}{2}\sin\theta d\theta$ ➡ $\frac{dP}{dJ} = ?$

角动量耦合的经典极限

归一化的几率密度为

$$dP(\theta) = \frac{1}{2}\sin\theta d\theta$$



$$\vec{J}^2 = \left(\vec{J}_1 + \vec{J}_2\right)^2 = J_1^2 + J_2^2 + 2J_1J_2\cos\theta$$

$$dJ^2 = 2JdJ = -2J_1J_2\sin\theta d\theta$$

$$JdJ = -J_1J_2\sin\theta d\theta$$

总角动量在$(J, J+dJ)$范围内的几率是

$$dP = \frac{J}{2J_1J_2}dJ$$

自旋1/2的两粒子
自旋角动量耦合

考虑两个自旋为 1/2 的粒子组成的系统（例如氢原子中的质子和电子或氦原子中的双电子等）的自旋波函数。设总自旋角动量 $\vec{S} = \vec{S}_1 + \vec{S}_2$。体系的希尔伯特空间是

$$\mathcal{H}_{12} = \mathcal{H}^1_{\text{ext}} \otimes \mathcal{H}^1_{\text{spin}} \otimes \mathcal{H}^2_{\text{ext}} \otimes \mathcal{H}^2_{\text{spin}}. \tag{8.5.32}$$

定义自旋空间的张量积是

$$\mathcal{H}_{\text{spin}} = \mathcal{H}^1_{\text{spin}} \otimes \mathcal{H}^1_{\text{spin}}. \tag{8.5.33}$$

因为每个自旋 1/2 粒子的自旋空间维度是 2，所以两个粒子的自旋空间维度是 4。因子化基矢为

$$|\sigma_1;\sigma_2\rangle \equiv |\sigma_1\rangle \otimes |\sigma_2\rangle, \tag{8.5.34}$$

具体形式如下：

$$\Big\{ |+;+\rangle,\quad |+;-\rangle,\quad |-;+\rangle,\quad |-;-\rangle \Big\}. \tag{8.5.35}$$

在 $\sigma_z^1 \otimes \sigma_z^2$ 表象中，

$$|+;+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad |+;-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\quad |-;+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad |-;-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.5.36}$$

选取力学量完备集是 $\{\hat{\vec{S}}^2, \hat{S}_z\}$。令 $|S,M\rangle$ 是 $\hat{\vec{S}}^2, \hat{S}_z$ 的共同本征态，

$$\begin{aligned}\hat{\vec{S}}^2|S,M\rangle &= S(S+1)\hbar^2|S,M\rangle \\ \hat{S}_z|S,M\rangle &= M\hbar|S,M\rangle.\end{aligned}$$

因为

$$\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z},$$

所以 $\hat{S}_z$ 算符的最大（最小）本征值和相应本征态是

$$M_{\max} = +\frac{1}{2} + \frac{1}{2} = +1, \qquad \text{本征态: } |+;+\rangle$$

$$M_{\min} = -\frac{1}{2} - \frac{1}{2} = -1, \qquad \text{本征态: } |-;-\rangle$$

将 $\hat{\vec{S}}^2$ 作用在 $|+;+\rangle$ 上就可得到本征值 S,

$$\begin{aligned}
\hat{\vec{S}}^2|+;+\rangle &= \left(\hat{S}_1^2+\hat{S}_2^2+2\vec{S}_1\cdot\vec{S}_2\right)|+;+\rangle \\
&= \left[\frac{3}{4}\hbar^2+\frac{3}{4}\hbar^2+2\frac{\hbar}{2}\frac{\hbar}{2}\left(\hat{\sigma}_{1x}\hat{\sigma}_{2x}+\hat{\sigma}_{1y}\hat{\sigma}_{2y}+\hat{\sigma}_{1z}\hat{\sigma}_{2z}\right)\right]|+;+\rangle \\
&= \left[\frac{3}{2}\hbar^2+2\frac{\hbar}{2}\frac{\hbar}{2}\right]|+;+\rangle \\
&= 2\hbar^2|+;+\rangle.
\end{aligned} \tag{8.5.43}$$

同理,

$$\hat{\vec{S}}^2|-;-\rangle=2\hbar^2|-;-\rangle, \tag{8.5.44}$$

这说明, $|+;+\rangle$ 和 $|-;-\rangle$ 都是 $\hat{\vec{S}}^2$ 算符的本征值 $S=1$ 的本征态。在耦合基矢中可以表示为

$$\begin{aligned}
\left|\frac{1}{2},\frac{1}{2},1,1\right\rangle_{\text{耦合基矢}} &= \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle_{\text{因子化基矢}}, \\
\left|\frac{1}{2},\frac{1}{2},1,-1\right\rangle_{\text{耦合基矢}} &= \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle_{\text{因子化基矢}}.
\end{aligned} \tag{8.5.45}$$

$$\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle=?$$

$$\hat{S}_- \left|\frac{1}{2},\frac{1}{2},1,1\right\rangle = \sqrt{2}\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle$$

$$\begin{aligned} & \left(\hat{S}_{1-}+\hat{S}_{2-}\right)|+;+\rangle = \left(\hat{S}_{1-}+\hat{S}_{2-}\right)|1+\rangle\otimes|2+\rangle \\ = \quad & \hat{S}_{1-}|1+\rangle\otimes|2+\rangle + |1+\rangle\otimes\hat{S}_{2-}|2+\rangle \\ = \quad & |1-\rangle\otimes|2+\rangle \ + \ |1+\rangle\otimes|2-\rangle \\ = \quad & |-;+\rangle \ + \ |+;-\rangle. \end{aligned}$$

$$\rightarrow \quad \left|\frac{1}{2},\frac{1}{2},1,0\right\rangle = \frac{1}{\sqrt{2}}\Big(|-;+\rangle \ + \ |+;-\rangle\Big)$$

最后一个态矢量可以利用和 $\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle$ 的正交性得到

$$\frac{1}{\sqrt{2}}\left(|+;-\rangle - |-;+\rangle\right)$$

$$S=0, S_z=0$$

$$\left|\frac{1}{2},\frac{1}{2},0,0\right\rangle$$

$$S=1 \quad : \quad \begin{pmatrix} |+;+\rangle \\ \frac{1}{\sqrt{2}}\left(|+;-\rangle+|-;+\rangle\right) \\ |-;-\rangle \end{pmatrix}$$

三重态，粒子
I和2的地位
是对称的

$$S=0 \quad : \quad \frac{1}{\sqrt{2}}\left(|+;-\rangle-|-;+\rangle\right).$$

单态，粒子
I和2的地位
是反对称的

$$2 \otimes 2 = 3 \oplus 1$$

矩阵表示

在 $\sigma_z^1 \otimes \sigma_z^2$ 表象中，

$$|+;+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+;-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-;+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-;-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\hat{S}_{1x} = \frac{\hbar}{2}\hat{\sigma}_x \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & \hat{I} \\ \hat{I} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\hat{S}_{2x} = \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_x & 0 \\ 0 & \hat{\sigma}_x \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$
\begin{aligned}
\hat{S}_{1y} &= \frac{\hbar}{2}\hat{\sigma}_y \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\hat{I} \\ i\hat{I} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\
\hat{S}_{2y} &= \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_y & 0 \\ 0 & \hat{\sigma}_y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\
\hat{S}_{1z} &= \frac{\hbar}{2}\hat{\sigma}_z \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{I}_2 = \frac{\hbar}{2}\begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\hat{S}_{2z} &= \hat{I}_2 \otimes \frac{\hbar}{2}\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} \hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

总自旋角动量算符在 $\sigma_z^1 \otimes \sigma_z^2$ 表象中的矩阵表示是

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}$$

$$\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \hat{S}^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

总角动量的升降算符

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\hat{S}^2|+;+\rangle = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{S}^2|-;-\rangle = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\hat{S}_-\left|\frac{1}{2},\frac{1}{2},1,1\right\rangle = \sqrt{2}\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle = \hat{S}_-|+;+\rangle = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left|\frac{1}{2},\frac{1}{2},1,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

最后一个态矢量 $|?\rangle$ 可以利用和 $\left|\frac{1}{2},\frac{1}{2},1,0\right\rangle$ 的正交性得到

$$|?\rangle = \frac{1}{\sqrt{2}}\Big(|-;+\rangle \ - \ |+;-\rangle\Big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$\hat{S}^2|?\rangle = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0,$$

$$\hat{S}_z|?\rangle = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0.$$

$$\left|\frac{1}{2},\frac{1}{2},0,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

从耦合基矢到因子化基矢之间的变换矩阵是

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

从因子化基矢到耦合基矢的变换矩阵为

$$\mathcal{S}' = \mathcal{S}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

总自旋角动量算符在耦合基矢空间的矩阵形式是

$$
\begin{aligned}
\hat{\vec{S}}^2_{\text{耦合基矢}} &= \langle SM|\hat{\vec{S}}^2\Big|S'M'\rangle \\
&= \langle SM\,|s_{1z};s_{2z}\rangle\Big\langle s_{1z};s_{2z}\Big|\hat{\vec{S}}^2\Big|s'_{1z};s'_{2z}\Big\rangle\langle s'_{1z};s'_{2z}\,\Big|S'M'\rangle \\
&= \mathcal{S}'\hat{\vec{S}}^2_{\text{因子化基矢}}\mathcal{S}'^{\dagger} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\
&= \hbar^2 \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}
$$

$$2 \otimes 2 = 3 \oplus 1$$

$$
\begin{aligned}
\hat{S}_z &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\
&= \hbar \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}
$$

$$
2 \otimes 2 = 3 \oplus 1
$$

C-G系数定义了一个幺正变换，将角动量的张量积空间化简为不可约化表示直和。

轨道-自旋
角动量耦合

Addition of Angular Momentum

$s = 1/2$ and $l = 1$

$$|\frac{1}{2},\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S_+ \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), I \otimes L_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$S_- \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right), I \otimes L_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$S_z \otimes I = \frac{\hbar}{2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right), I \otimes L_z = \hbar \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right).$$

$$J_{\pm,z} = S_{\pm,z} \otimes I + I \otimes L_{\pm,z}$$

$$J_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

$$J_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array} \right), \qquad J_z = \hbar \left(\begin{array}{ccc|ccc} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array} \right).$$

$$|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \qquad |\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right).$$

$$J_+|\frac{3}{2},\frac{3}{2}\rangle \;=\; J_-|\frac{3}{2},-\frac{3}{2}\rangle \;=\; 0$$

$$J_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad J_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$J_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle$$

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ \sqrt{2} \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle, \qquad J_+|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle$$

$$J_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right), \qquad |\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$$

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = J_-|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,1\rangle$$

$$= |\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,1\rangle + \sqrt{2}|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle,$$

$$J_+|\frac{3}{2},-\frac{3}{2}\rangle = J_+|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,-1\rangle$$

$$= |\frac{1}{2},\frac{1}{2}\rangle \otimes |1,-1\rangle + \sqrt{2}|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle$$

Their orthogonal linear combinations belong to the $j = 1/2$ representation

$$|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = J_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0, \text{ and } J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$J_- = \hbar\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right),$$

Define a unitary rotation

$$U = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 \end{array}\right)$$

$$U|\frac{3}{2},\frac{3}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, U|\frac{3}{2},\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, U|\frac{3}{2},-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, U|\frac{3}{2},-\frac{3}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix},$$

$$U|\frac{1}{2},\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}, U|\frac{3}{2},\frac{3}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{pmatrix}.$$

Generators in new basis

$$V_{1/2} \otimes V_1 = V_{3/2} \oplus V_{1/2}$$

$$2 \otimes 3 = 4 \oplus 2$$

$$UJ_zU^\dagger = \hbar \left(\begin{array}{cccc|cc} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{array}\right),$$

$$UJ_+U^\dagger = \hbar \left(\begin{array}{cccc|cc} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$UJ_-U^\dagger = \hbar \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

C-G系数定义了一个幺正变换，将角动量的张量积空间化简为不可约化表示直和。