粒子物理

15. 夸克模型

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Gell-mann 的八正法

*) 同位旋和奇异数构成新的有用的分析工具
(SU(2))

*) 利用群论概念，寻找强相互作用的更高对称性，可以将不同同位旋和奇异数的粒子包容进来

⇒ 1961年 Gell-mann 和 Ne'eman 提出 SU(3) 群

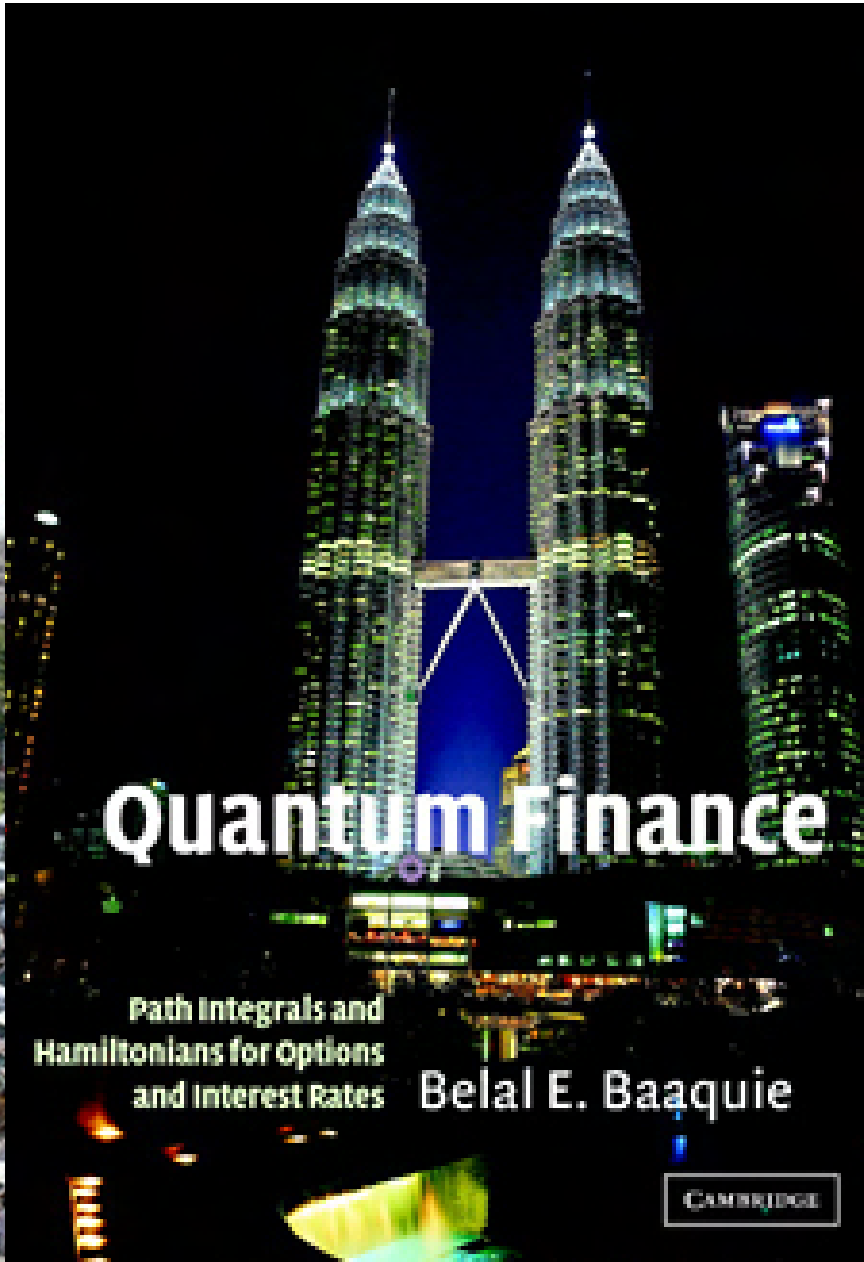
注意：他们原始动机并不是为了强子分类，而是要试图建立一个具体的强相互作用量子场论

此时量子场论正迅速衰败，被视作是完全过时的

⇒ 直接导致 SU(3) 提出不久就与其规范理论之根源分离，变为一种自生自灭的粒子分类方法

为什么QFT在走下坡路?

今天量子场论
如日中天
炙手可热



量子电动力学（QED）

拉格朗日量:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - qA_{\mu}\bar{\psi}\gamma^{\mu}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

可精确求解　　　　不可精确求解

微扰求解

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$

n个光子贡献α^n

量子力学二阶微扰项

$$E_a^{(2)} = \sum_{i \neq a} \frac{\left|\left\langle \psi_i^{(0)} | \hat{H}_I | \psi_a^{(0)} \right\rangle\right|^2}{E_a^{(0)} - E_i^{(0)}} = \sum_{i \neq a} \frac{1}{E_a^{(0)} - E_i^{(0)}} \left\langle \psi_a^{(0)} | \hat{H}_I | \psi_i^{(0)} \right\rangle \left\langle \psi_i^{(0)} | \hat{H}_I | \psi_a^{(0)} \right\rangle$$

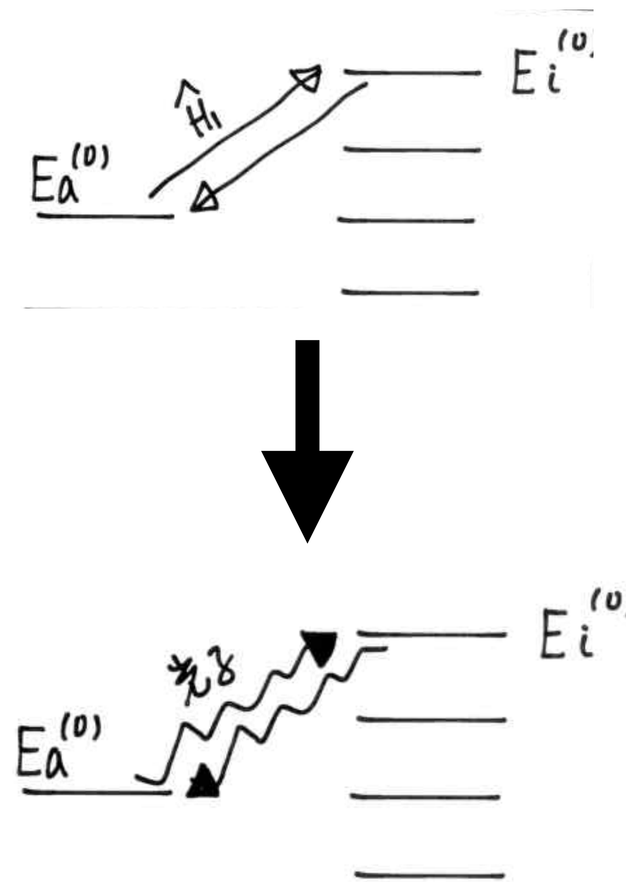
收敛性：

1) $\left\langle \psi_i^{(0)} | \hat{H}_I | \psi_a^{(0)} \right\rangle$很大，导致对各态求和不收敛

——> 紫外发散

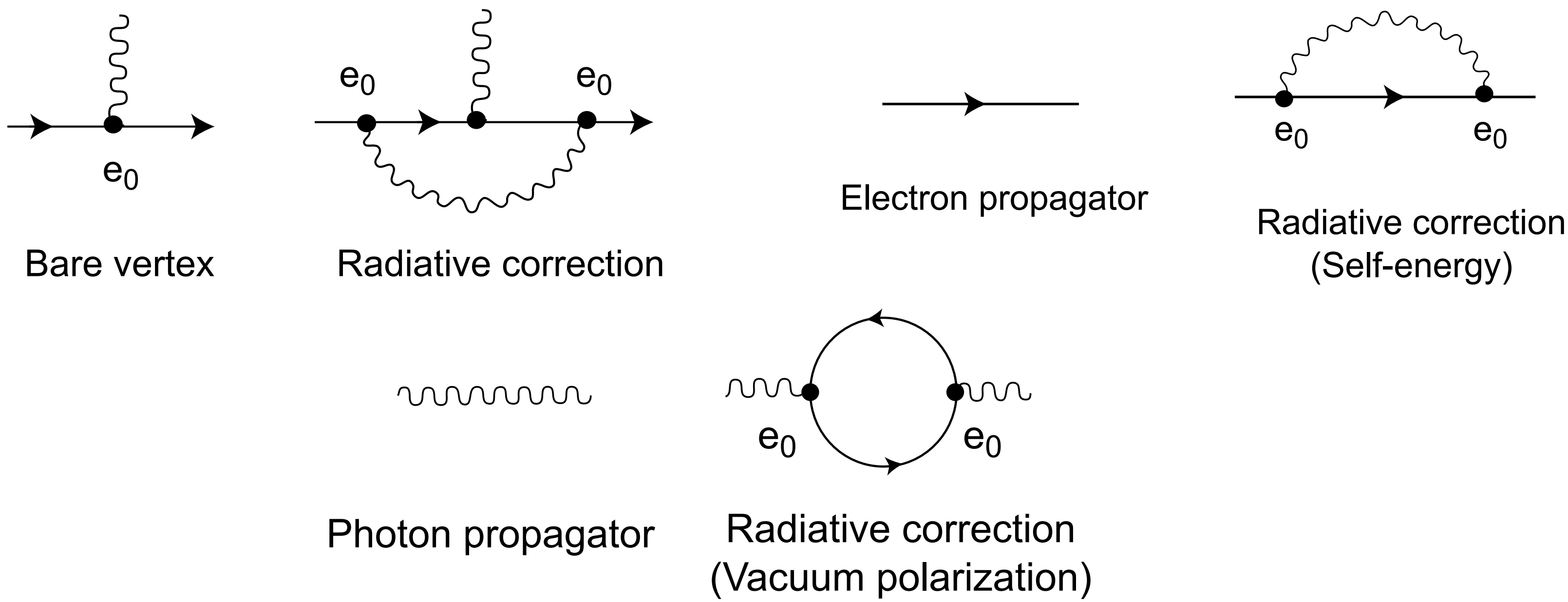
2) E_a能级附近存在许多（或连续的）能级满足 $|E_i - E_a| \sim 0$ 从而导致对各态求和不收敛

——> 红外发散



重整化

QED: 微扰展开计算中的无穷大问题



“整个30年代，物理学界共识是，量子场论并不被看好。它可能有用，但只是权宜之计，需要添加全新的东西才能使它说的通。”

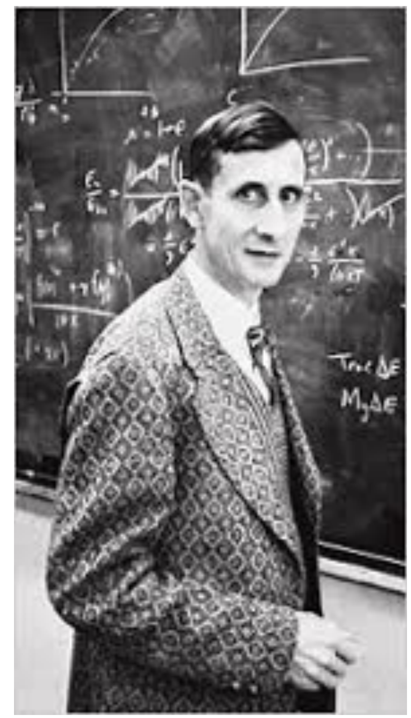
QED重整化

- 20世纪40年代后期才消除QED理论中的不健全之处

  Feynman, Schwinger, Tomonaga分别提出重整化思想

  1949年Dyson证明他们三种方案是等价的

1965 Nobel



Dyson

Freeman Dyson

Note: 戴森（Freeman Dyson）早年在剑桥大学追随著名的数学家 G.H. 哈代研究数学，1945 年获得数学系的学士学位后，于 1947 年到美国康奈尔大学跟随汉斯·贝特和理查德·费曼学习。他证明了施温格和朝永振一郎发展的变分法方法和费曼的路径积分法的等价性，为量子电动力学的建立做出了决定性的贡献。1949 年戴森提出 Dyson series，这一工作启发 Ward 研究并提出 Ward 等式。

戴森没有博士学位，但由于他的杰出贡献，康奈尔大学于 1951 年聘请戴森为物理学教授。这在今天是难以想象的。戴森获得很多荣誉学位，其中包括 Yeshiva University (1966), University of Glasgow (1974), Princeton University (1974), University of York (1980), City University of London (1981), New School of Social Research (1982), Rensselaer Polytechnic (1983), Susquehanna University (1984), Depauw University (1987), Rider College (1989), Bates College (1991), Haverford College (1991), Dartmouth College (1995), Federal Inst. of Tech. (ETH), Switzerland (1995), Scuola Normale Superiore, Pisa, Italy (1996), University of Puget Sound (1997), Oxford University (1997), Clarkson University (1998), Rockefeller University (2001), St. Peter's College (2004), Georgetown University (2005), University of Michigan (2005), University of the Sciences (2011)。

Muon g-2

Kinoshita

$$
\begin{aligned}
\tfrac{1}{2} g_{\text{theory}} = \; & 1 && \text{(a) 1928 (Dirac equation)} \\
& + (\alpha/2\pi) && \text{(b) 1949 (1 diagram)} \\
& -0.32848\,(\alpha/\pi)^2 && \text{(c) 1958 (18 diagrams)} \\
& + (1.195 \pm 0.026)\,(\alpha/\pi)^3 && \text{(d) 1974 (72 diagrams)} \\
& - (1.7283\,(35))\,(\alpha/\pi)^4 + (\text{Non-QED}) && \text{(e) 2006 (891 diagrams).}
\end{aligned}
$$

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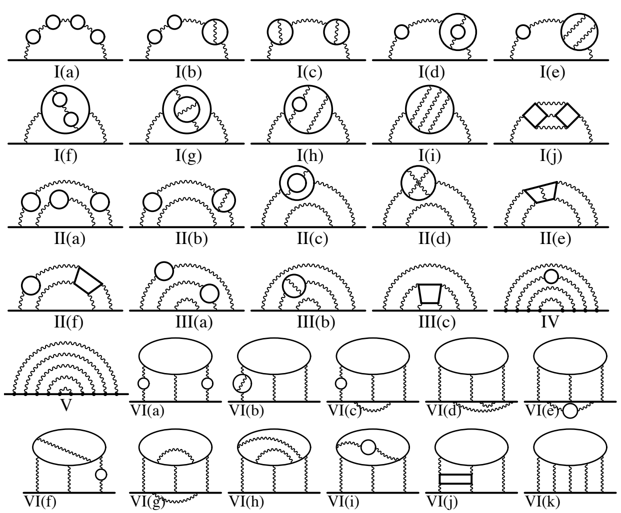
Complete Tenth-Order QED Contribution to the Muon $g-2$

Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]



5圈图（总计12672个费曼图）

计算精度： 10^{-12}

人类精确计算的登峰造极之作

量子场论大萧条

1949年后的几年内，因为QED理论的极大成功，人们对量子场论的热情处于发烧状态。许多理论物理学家都认为很快就会完全理解所有的微观现象，不仅仅限于光子、电子和正电子而已。

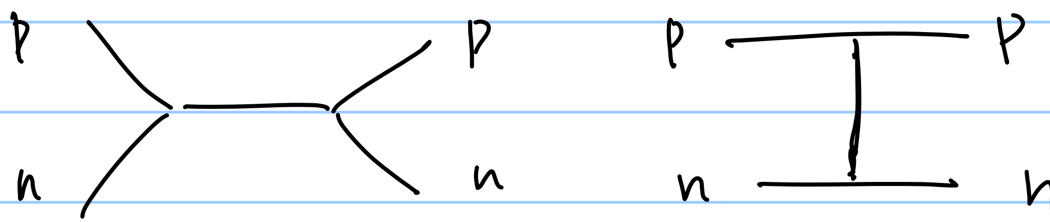
然而不久，这种信心就崩溃了——量子场论的股票在物理学股市上大跌，并因此进入第二轮熊市。不幸的是，这次大萧条持续了近20年。

B) 量子场论用于弱相互作用 → "幺正性灾难"
(破坏几率守恒性)

原因: W/Z 粒子未曾发现

C) 量子场论用于强相互作用 → 失败更直接和显著

1935年 Yukawa 仿效 QED 建立一个有效理论, (重整化没有问题)



虽然可重整化, 并且遵从守恒定律, 但这个理论没有什么用!!!

因为, 无法使用微扰论计算 ($g_{pn}^2 \simeq g_{nn}^2 \simeq g_{pp}^2 \simeq 15$)

⟹ 在Yukawa强相互作用理论中，高阶扰动大于低阶贡献
为了得到有意义的理论预言，必须对无限多项的序列求和
但无人知道怎么做！ ⟶ 失败！

所以上个世纪50年代中期，场论方法不被人看好！
希望越大，失望也越大！（SUSY）

*) S-矩阵 (理论家是非常聪明的)

核心问题：强子过程能否用微扰展开级数的费曼图来描述？

⟹ 如果能，那么如何对发散级数求和？

换种思路：Yukawa 理论的 Pion 被实验验证了，
如果 Yukawa 理论是正确的，那么有意义的一定
求和之后的结果，而级数中个别项不具有重要的
物理意义。

Observed: $|i\rangle \to |f\rangle$

并未曾看到中间传播的核子或 Pion

解决方法：放弃 Feynman 图，仅研究跃迁概率本身 (S-矩阵)

1956-1959年 CHEW等人得到：

S-矩阵可视为有关变量的解析函数

人们仍然用基于费曼图的微扰论做计算，但仅仅将其作为分析S矩阵解析性质的启发性工具。

激进的CHEW宣称QFT在处理强相互作用已经死亡。

⟶ 提出"bootstrap"理论。

靴袢理论：

从S矩阵的解析结构导出一组由无穷多个耦合的非线性方程构成的方程组，但无人知道如何求解。

CHEW指出：此方程组有解，而且解是唯一的，

并认为通过自洽性要求，此解决定所有强子的属性。

⟹ 所有强子的地位都是均等的，通过 bootstrap

使自己成为S矩阵方程自洽解的一部分。

例如：将无穷多个方程实施截断，输入π性质

⟹ 自洽地标出ρ介子属性

用同位旋输入 ⟹ SU(3)强子对称性

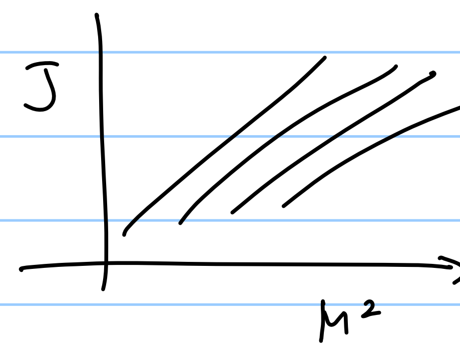
雷奇理论：

Reggie 非相对论势能散射，可采用复数的能量和角动量变量，
而不是用能量和动量转移，来分析解析性质.

CHEW 将之推广到相对论性 S矩阵方法

*) 高能量和小动量转移条件下，S矩阵的行为可按
由量的 Reggie 奇点的性质来理解
这些 Reggie 奇点是自旋依赖于能量的准粒子

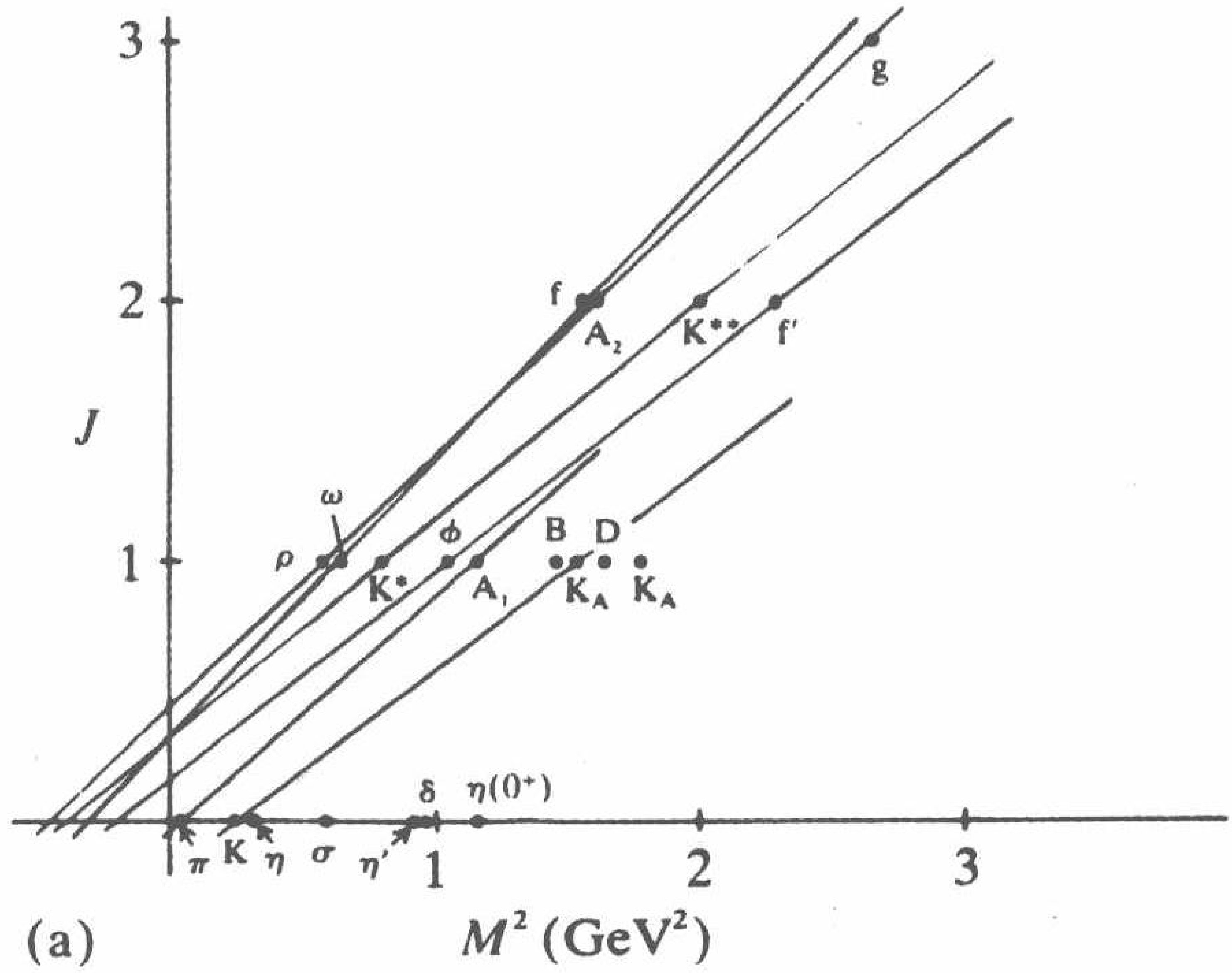
*) Reggie 奇点的能量达到其自旋取整数或半整数时
⟹ 可观测的粒子



*) 全新粒子分类
按 Reggie 轨迹上位置来分类

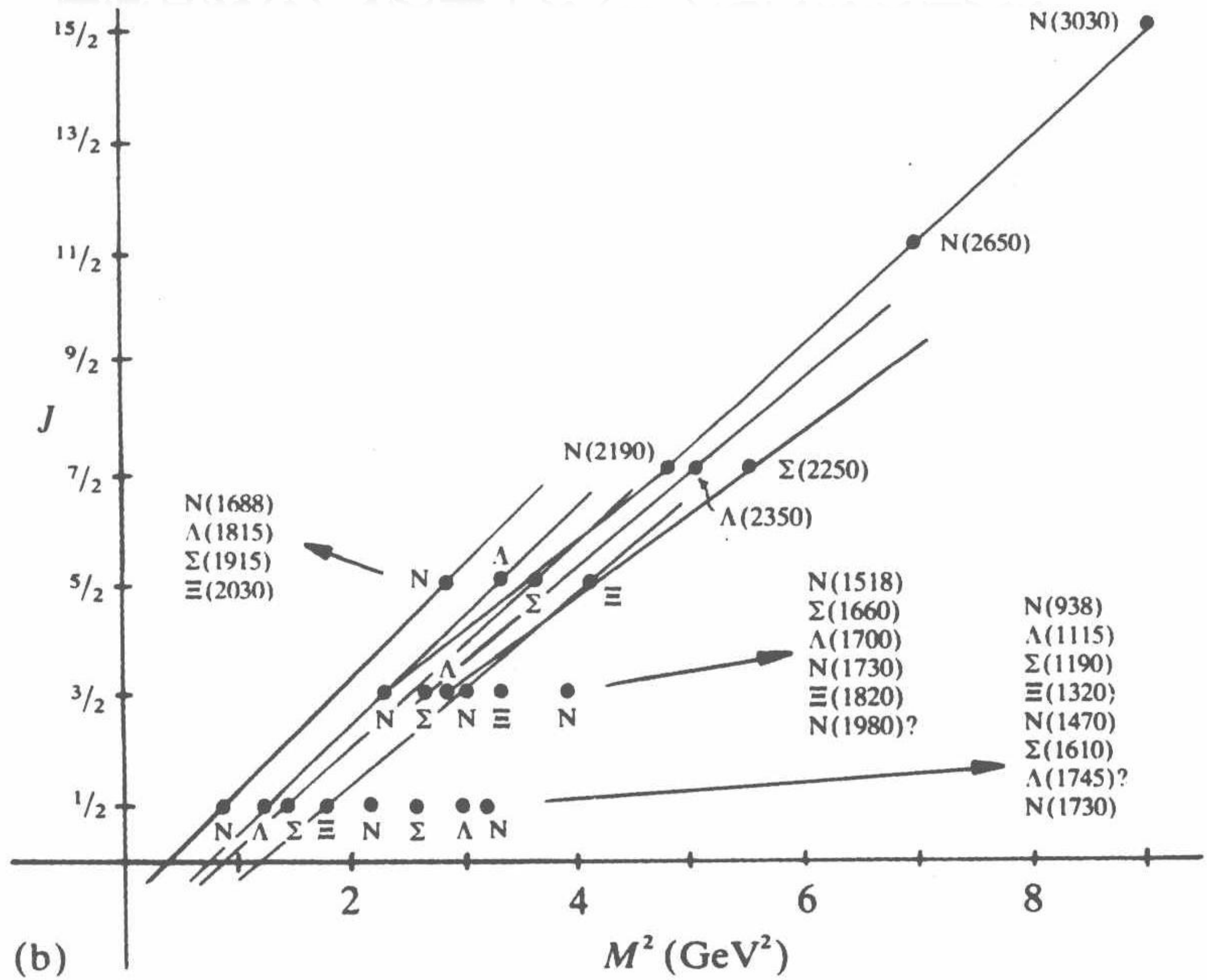
不同强子的雷吉轨迹：(a) 介子，

注意自旋(J)与质量平方(M^2)之间的线性关系



不同强子的雷吉轨迹：(b) 重子

注意自旋(J)与质量平方(M^2)之间的线性关系



夸克模型

标题 2014/11/17

1964年 Gell-mann 提出

如果八正法是正确的，那么其背后一定隐藏着未知的规律。

⟹ 强子为复合粒子，并认为这些粒子为更基本的且具有SU(3)对称性

方案1：强子有四种基本实体组成，每个实体的电荷为0或1

好处：可以重复出SU(3)多重强子结构

坏处：其中1个粒子起到基本量子作用，与其余3个不同

方案2：允许电荷以非整数形式存在，则可得到更简单，更优美的结构 ⟶ 夸克模型

与此同时 Zweig 独立提出夸克模型，他将之命名为 ACE。

(这场冠命大战最终以 Gell-mann 大获全胜而告终。)

Gell-mann 和 Zweig 是根据完全不同的动机来得到"夸克"的概念。

共同点1：强子由不同的内部成分构成，其性质取决于这些成分的性质

Quark: $s=1/2$，$b=\frac{1}{3}$

	up	down	strange
	$I=1/2$	$1/2$	0
	$S=0$	0	-1
	$Q=\frac{2}{3}$	$-1/3$	$-1/3$

复合粒子的想法并不是革命性的

1949. Yang 和 Fermi 提出 π 是核子-反核子组成

1956 Sakata 提出坂田模型（p, n 和 Λ）→ 为解释奇异数

1964. Ω^- 粒子被发现，排除了 Sakata model

（在 Sakata model 中 $\Omega^- = \Omega^-(\Lambda\Lambda\Lambda)$）

共同点2：强子态可以归纳入SU(3)群1, 3, 6, 8, 10, 27等组元数目的多重态

共同点3：都无法解释为何未观测到夸克

复合粒子意味着其内部成分可以被观测（原子：电子和原子核）

Gell-mann在文中注明：

由于电荷数和重子数守恒，必有一种夸克是绝对稳定的

地球表面物质将带着稳定夸克，但有可能浓度积量非常微小，

以及于无法观测到

⇒ 探测以寻找 Q = 2/3 或 -1/3 的稳定夸克

（今天的暗物质也有可能是这样的）

*) 虽然 Gell-mann 自己本人都怀疑夸克，但实验家却非常喜欢。

- 寻找核时代 ⟹ 电荷必定是整数的 (挑战)
- 三类实验：① 加速器 ② 宇宙线 ③ 全新版"油滴"实验 } 来找到分数电荷

*) 虽然并未直接测到夸克，但用夸克或组分夸克来解释强子分类却取得了极大成功！

这并不奇怪！物理是实验科学。任何理论都是对自然界的一种描述。只要可以解释实验，那就是好的有效描述。

*) Zweig的组分夸克模型 (Composite Quark model ≡ CQM)

(1937生于Moscow, 1959年毕业于University of Michigan数学系)

作为理论物理新手，没有太多框架约束，敢想敢干

高能物理学界的前辈们并不认同CQM，甚至有人宣称Zweig的CQM是"骗人"的工作 —— 这比民科还狠！

为什么会这样？

上个世纪60年代，强相互作用的两个框架

量子场论 和 S-矩阵

两者都无法接受CQM

① QFT：无法直接观测到组分夸克 ⟶ m_Q 非常大 ~ GeV

$\langle Q\bar{Q}\rangle \sim m_\pi \ll 2m_Q$ 要求结合能非常大

$\Rightarrow E^{\text{结合能}}_{QQ} \sim m_Q$

组分夸克间相互作用太强，无法计算

② S-矩阵：更加反对。因为 S-矩阵理论中，所有强子的地位都是相等的。

bootstrap：根本不相信有什么基本组分。

当然，最重要是实验上未曾观测到组分夸克!!

但人们承认 CQM 的巨大的启发价值。

⟹ 唯象研究取得了极大的成功

（将抽象程度较低的模型应用于数据分析或进一步指导实验）

*) Gell-mann 的夸克和流代数

上世纪 60 和 70 年代压在场论头上的两座大山

(1) 高能领域的散射的 Regge 理论

(2) 低能领域的有关束缚态的 CQM

⇒ "流代数" 保留了场论革命的火种

Gell-mann 是在流代数基础上讨论夸克，

└ 但说服物理学家接受夸克的概念

Gell-mann 文章仅 2 页，唯一的公式是计算自由夸克的矢量"弱流"的对易子

1958年 Gell-mann和Feynman给弱相互作用唯象的V-A理论

以QED为基本，将弱相互作用类比于QED中电磁流之间

发生的电磁相互作用

问题：纯轻子的弱和电磁相互作用非常清楚，

弱流和电磁流可以通过场来构造

但实验上找到成百上千的强子？

Gellmann假设各种强子流之间的关系可以像轻子流的

V-A理论一样模型化

轻子理论 ⟶ 流产生SU(2)代数

强子理论 ⟶ 强子流产生SU(3)代数　　SU(3) × SU(3)
　　　　　　　　　　　　　　　　　　　　矢量流　轴矢流

⟹ 流代数名字的来源

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks:

★ Assume the strong interaction treats all quark flavours equally (it does)

- Because $m_u \approx m_d$:

  The strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- Choose the basis

$$u = \begin{pmatrix}1\\0\end{pmatrix} \qquad d = \begin{pmatrix}0\\1\end{pmatrix}$$

- Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space

$$\begin{pmatrix}u'\\d'\end{pmatrix} = \hat{U}\begin{pmatrix}u\\d\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}u\\d\end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger\hat{U} = 1$

➡ **8 – 4 = 4 independent matrices**

- In the language of group theory the four matrices form the U(2) group

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\boxed{\det U = 1}$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

$$\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$$

- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isopin

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of **total isospin** I and the third component of isospin I_3

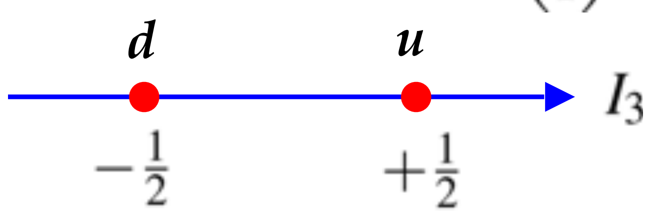
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

  with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\,\tfrac{1}{2}, +\tfrac{1}{2}\,\rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



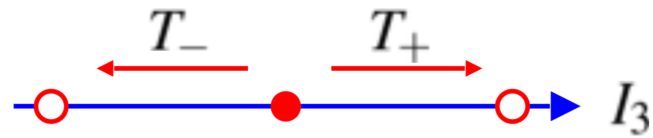
$$I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$$T_- \equiv T_1 - iT_2$$

u → d



$$T_+ \equiv T_1 + iT_2$$

d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\,|I,I_3+1\rangle$$

$$T_-|I,I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\,|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$T_+ u = 0 \qquad T_+ d = u \qquad T_- u = d \qquad T_- d = 0$$

- Ladder operators turn $u \rightarrow d$ and $d \rightarrow u$

★ Combination of isospin: **e.g.** what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \rightarrow |I, I_3\rangle$$

- I_3 additive : $I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

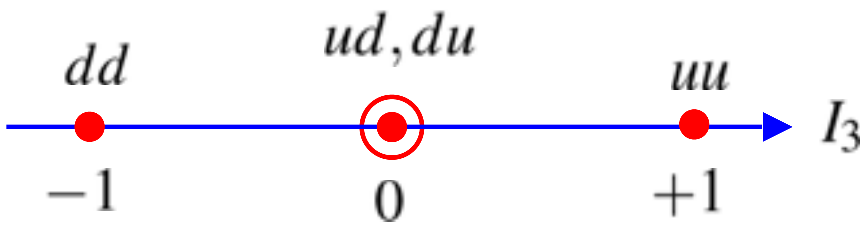
- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

Combining Quarks

Goal: derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that **fermion wave-functions are anti-symmetric**

Isospin starts to become useful in defining states of more than one quark.
e.g. two quarks, here we have four possible combinations:



Note: ⦿ represents two states with the same value of I_3

- We can immediately identify the extremes (I_3 additive)

$$uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle \qquad dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

To obtain the $|1,0\rangle$ state use ladder operators

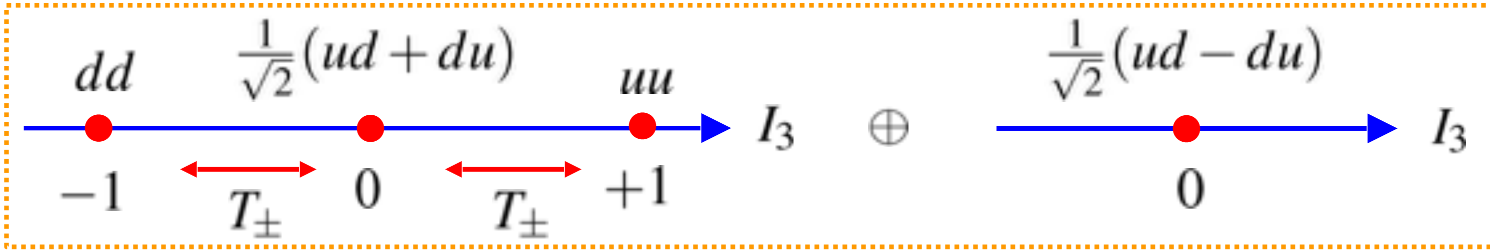
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

The final state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

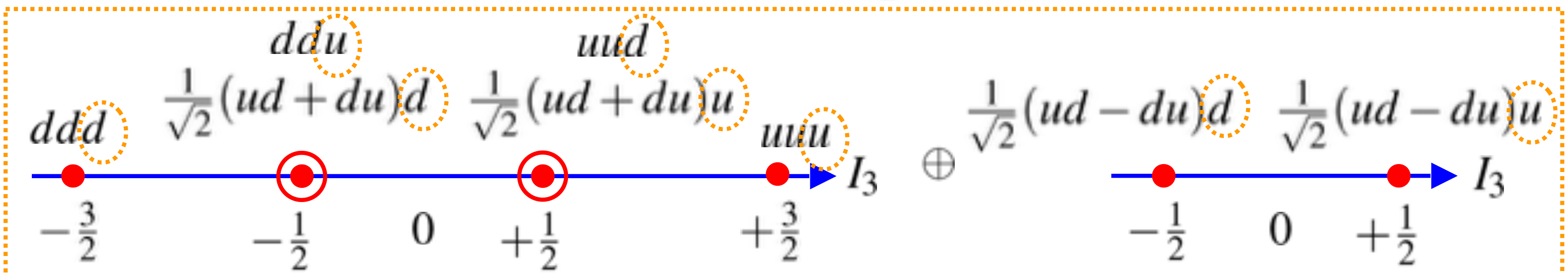
$$\rightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud - du)$$

- From four possible combinations of isospin doublets obtain a **triplet** of **isospin 1** states and a **singlet** **isospin 0** state $2 \otimes 2 = 3 \oplus 1$



- Can move around <u>within</u> **multiplets** using ladder operators
- note, as anticipated $I_3 = \frac{1}{2}(N_u - N_d)$
- States with different **total isospin** are physically different – the **isospin 1 triplet** is **symmetric** under interchange of quarks 1 and 2 whereas **singlet** is **anti-symmetric**

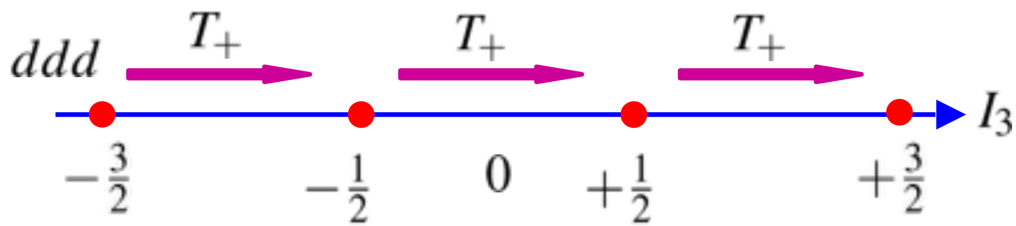
★ Now add an additional up or down quark. From **each of the above 4 states** get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$



6 **2**

- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from ddd

★**Derive the $I=\frac{3}{2}$ states from $ddd \equiv |\frac{3}{2},-\frac{3}{2}\rangle$**



$$
\begin{aligned}
T_+|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= T_+(ddd)=(T_+d)dd+d(T_+d)d+dd(T_+)d \\
\sqrt{3}|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= udd+dud+ddu \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(udd+dud+ddu) \\
T_+|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(udd+dud+ddu) \\
2|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+uud+duu+udu+duu) \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+duu) \\
T_+|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(uud+udu+duu) \\
\sqrt{3}|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uuu+uuu+uuu) \\
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= uuu
\end{aligned}
$$

★ **From the 6 states on previous page, use orthoganality to find $|\frac{1}{2},\pm\frac{1}{2}\rangle$ states**

★ **The 2 states on the previous page give another $|\frac{1}{2},\pm\frac{1}{2}\rangle$ doublet**

★The eight states $uuu, uud, udu, udd, duu, dud, ddu, ddd$ are grouped into an isospin quadruplet and two isospin doublets

$$2\otimes2\otimes2=2\otimes(3\oplus1)=(2\otimes3)\oplus(2\otimes1)=4\oplus2\oplus2$$

- Different multiplets have different symmetry properties

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = uuu$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(uud+udu+duu)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(ddu+dud+udd)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = ddd$$

S — A quadruplet of states which are symmetric under the interchange of <u>any</u> two quarks

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2uud-udu-duu)$$

$\mathbf{M_S}$ — Mixed symmetry. Symmetric for 1 ⟷ 2

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udu-duu)$$

$\mathbf{M_A}$ — Mixed symmetry. Anti-symmetric for 1 ⟷ 2

- Mixed symmetry states have no definite symmetry under interchange of quarks $1\leftrightarrow3$ etc.

Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = \uparrow\uparrow\uparrow$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

S — A quadruplet of states which are symmetric under the interchange of any two quarks

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_S — Mixed symmetry. Symmetric for 1 ⟷ 2

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_A — Mixed symmetry. Anti-symmetric for 1 ⟷ 2

- Can now form total wave-functions for combination of three quarks

Baryon Wave-functions (ud)

★Quarks are fermions so require that the total wave-function is **anti-symmetric** under the interchange of any two quarks

★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

★ The colour wave-function for all bound qqq states is **anti-symmetric**

- Here we will only consider the lowest mass, ground state, baryons where there is no internal orbital angular momentum.
- For L=0 the spatial wave-function is **symmetric** (-1)^L.

➡ $\xi_{\text{colour}}\eta_{\text{space}}$ — anti-symmetric

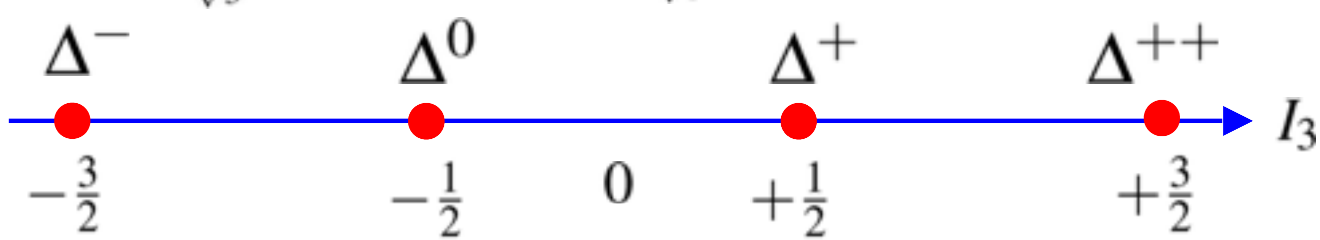
➡ $\phi_{\text{flavour}}\chi_{\text{spin}}$ — symmetric

} Overall anti-symmetric

★ Two ways to form a totally symmetric wave-function from spin and isospin states:

❶ combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

ddd $\quad \frac{1}{\sqrt{3}}(ddu+dud+udd)$ $\quad \frac{1}{\sqrt{3}}(uud+udu+duu)$ $\quad uuu$



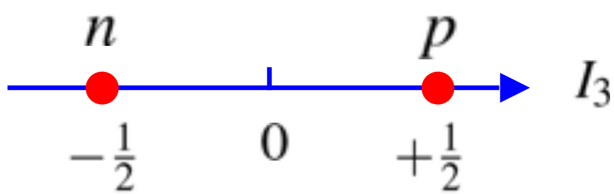
Spin 3/2
Isospin 3/2

❷ combine mixed symmetry spin and mixed symmetry isospin states

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is totally symmetric (i.e. symmetric under $1 \leftrightarrow 2;\ 1 \leftrightarrow 3;\ 2 \leftrightarrow 3$)



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

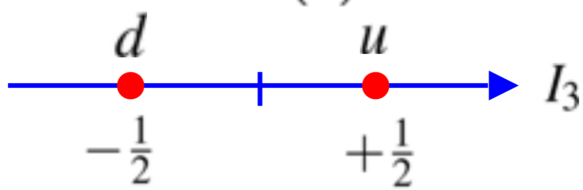
$$\begin{aligned}|p\uparrow\rangle = \frac{1}{\sqrt{18}}(\ & 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ & 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + \\ & 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\uparrow)\end{aligned}$$

NOTE: not always necessary to use the fully symmetrised proton wave-function, **e.g.** the first 3 terms are sufficient for calculating the proton magnetic moment

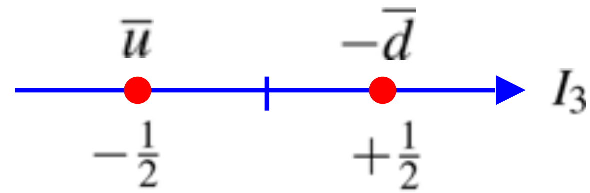
Anti-quarks and Mesons (u and d)

★The u, d quarks and ū, d̄ anti-quarks are represented as isospin doublets

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$



$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$



$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **Subtle point:** The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d;\ \bar{u} \leftrightarrow \bar{d}$

- Consider the effect of ladder operators on the anti-quark isospin states

e.g

$$T_+\bar{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

- The effect of the ladder operators on anti-particle isospin states are:

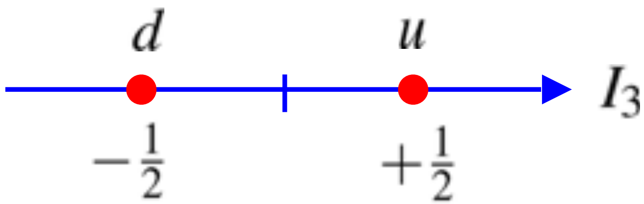
$$T_+\bar{u} = -\bar{d} \qquad T_+\bar{d} = 0 \qquad T_-\bar{u} = 0 \qquad T_-\bar{d} = -\bar{u}$$

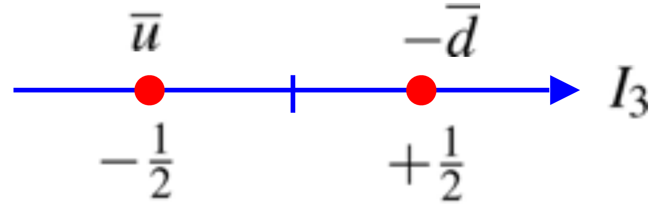
Compare with

$$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

Light ud Mesons

★ Can now construct meson states from combinations of up/down quarks





- Consider the $q\overline{q}$ combinations in terms of isospin

$$|1,+1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle} = -u\overline{d}$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle} = d\overline{u}$$

The bar indicates this is the isospin representation of an anti-quark

To obtain the $I_3 = 0$ states use ladder operators and orthogonality

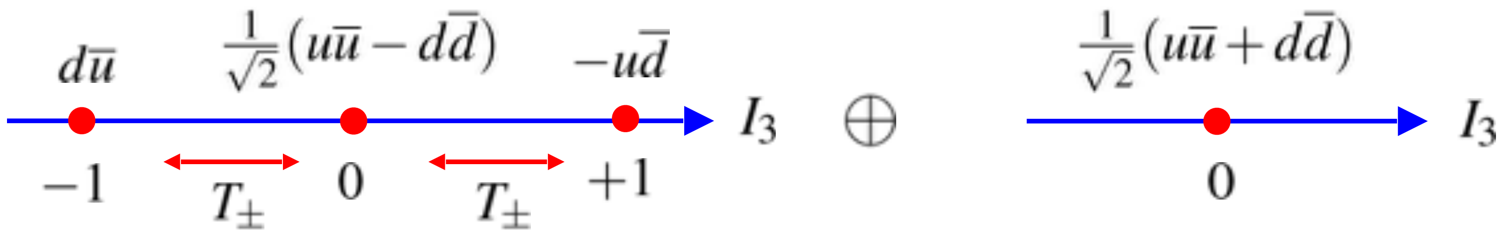
$$\begin{aligned} T_-|1,+1\rangle &= T_-\left[-u\overline{d}\right] \\ \sqrt{2}|1,0\rangle &= -T_-\left[u\right]\overline{d} - uT_-\left[\overline{d}\right] \\ &= -d\overline{d} + u\overline{u} \end{aligned}$$

$\Rightarrow$ $$|1,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right)$$

- Orthogonality gives: $$|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$$

★ **To summarise:**

Triplet of $I=1$ states and a singlet $I=0$ state



- **You will see this written as** $2 \otimes \bar{2} = 3 \oplus 1$

Quark doublet → 2; **Anti-quark doublet** → $\bar{2}$

- **To show the state obtained from orthogonality with $|1,0\rangle$ is a singlet use ladder operators**

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) = \frac{1}{\sqrt{2}}\left(-u\bar{d}+u\bar{d}\right) = 0$$

similarly $\quad T_-|0,0\rangle = 0$

★ **A singlet state is a "dead-end" from the point of view of ladder operators**

SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an <u>exact symmetry</u>. But m_s not so very different from m_u, m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★In SU(3) **uds** flavour symmetry contains SU(2) **ud** flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

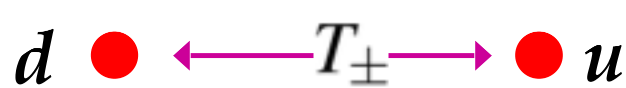
i.e. u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$

  with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$



- Now consider the matrices corresponding to the u ↔ s and d ↔ s

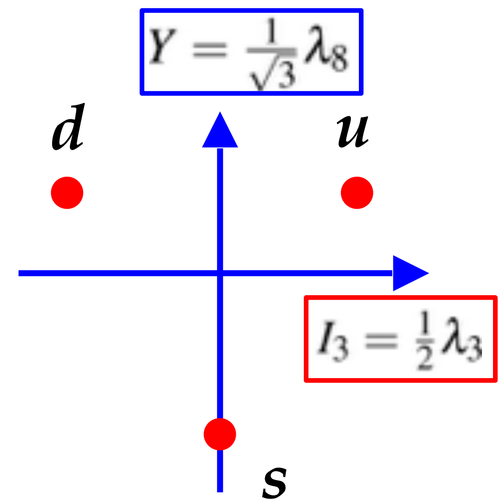
u ↔ s $\quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d ↔ s $\quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices

- However the three diagonal matrices are not be independent.
- Define the eighth matrix, λ_8, as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

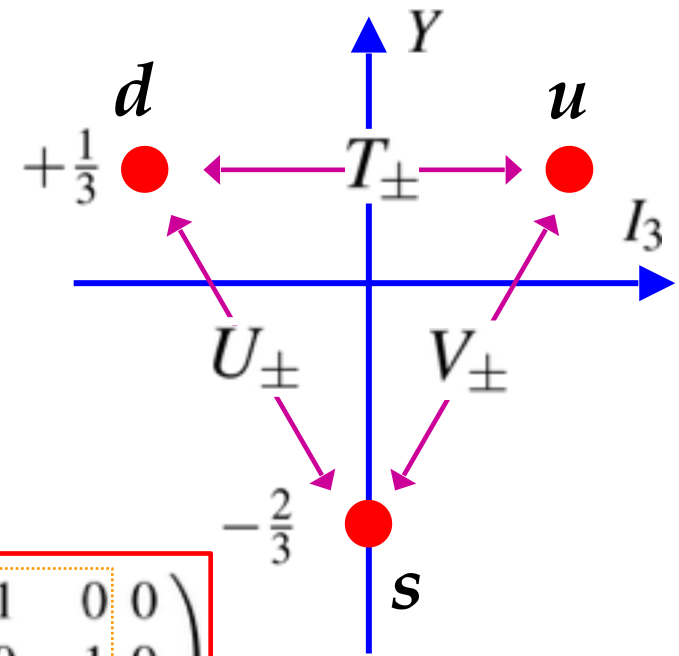


which specifies the "vertical position" in the 2D plane

"Only need two axes (quantum numbers) to specify a state in the 2D plane": (I_3, Y)

★ **The other six matrices form six ladder operators which step between the states**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



with $I_3 = \tfrac{1}{2}\lambda_3$ $\quad Y = \tfrac{1}{\sqrt{3}}\lambda_8$

and the eight Gell-Mann matrices

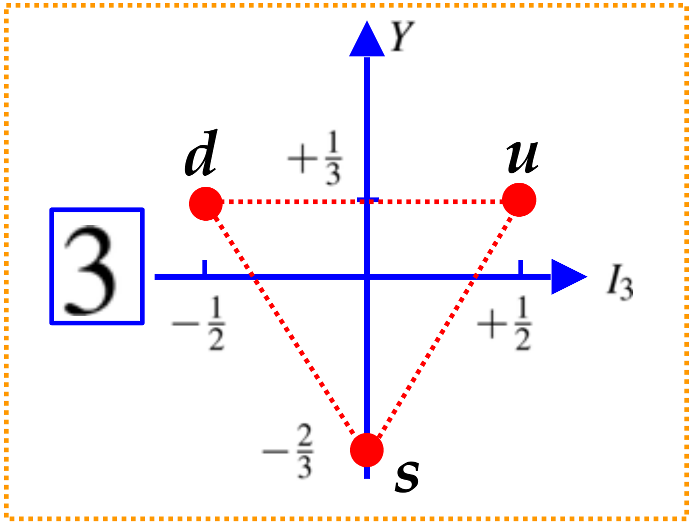
u ↔ d $\quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

u ↔ s $\quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

d ↔ s $\quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quarks and anti-quarks in SU(3) Flavour

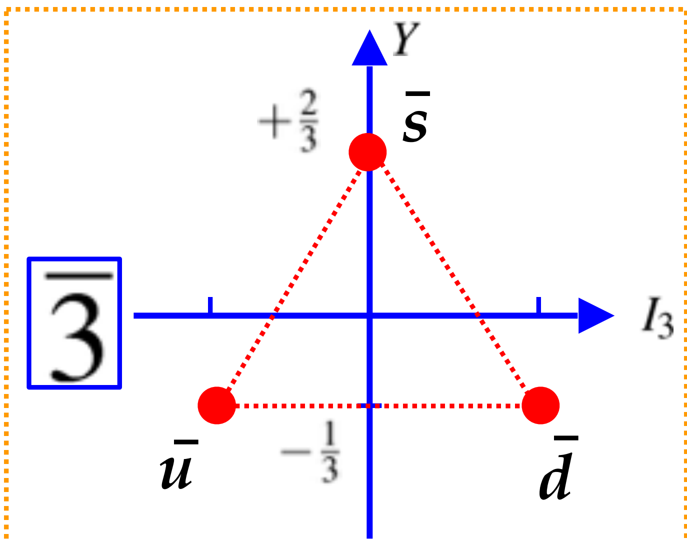


Quarks

$$I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0$$

$$Yu = +\tfrac{1}{3}u; \quad Yd = +\tfrac{1}{3}d; \quad Ys = -\tfrac{2}{3}s$$

•The anti-quarks have opposite **SU(3)** flavour quantum numbers



Anti-Quarks

$$I_3 \bar{u} = -\tfrac{1}{2}\bar{u}; \quad I_3 \bar{d} = +\tfrac{1}{2}\bar{d}; \quad I_3 \bar{s} = 0$$

$$Y\bar{u} = -\tfrac{1}{3}\bar{u}; \quad Y\bar{d} = -\tfrac{1}{3}\bar{d}; \quad Y\bar{s} = +\tfrac{2}{3}\bar{s}$$

SU(3) Ladder Operators

- SU(3) *uds* flavour symmetry contains *ud*, *us* and *ds* SU(2) symmetries

- Consider the $u \leftrightarrow s$ symmetry "V-spin" which has the associated $s \longrightarrow u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

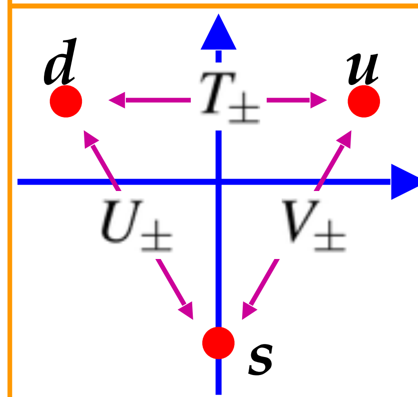
★The effects of the six ladder operators are:

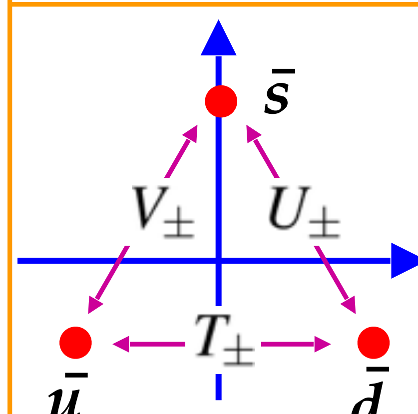
$T_+ d = u; \quad T_- u = d;$	$T_+ \bar{u} = -\bar{d}; \quad T_- \bar{d} = -\bar{u}$
$V_+ s = u; \quad V_- u = s;$	$V_+ \bar{u} = -\bar{s}; \quad V_- \bar{s} = -\bar{u}$
$U_+ s = d; \quad U_- d = s;$	$U_+ \bar{d} = -\bar{s}; \quad U_- \bar{s} = -\bar{d}$

all other combinations give zero

SU(3) LADDER OPERATORS

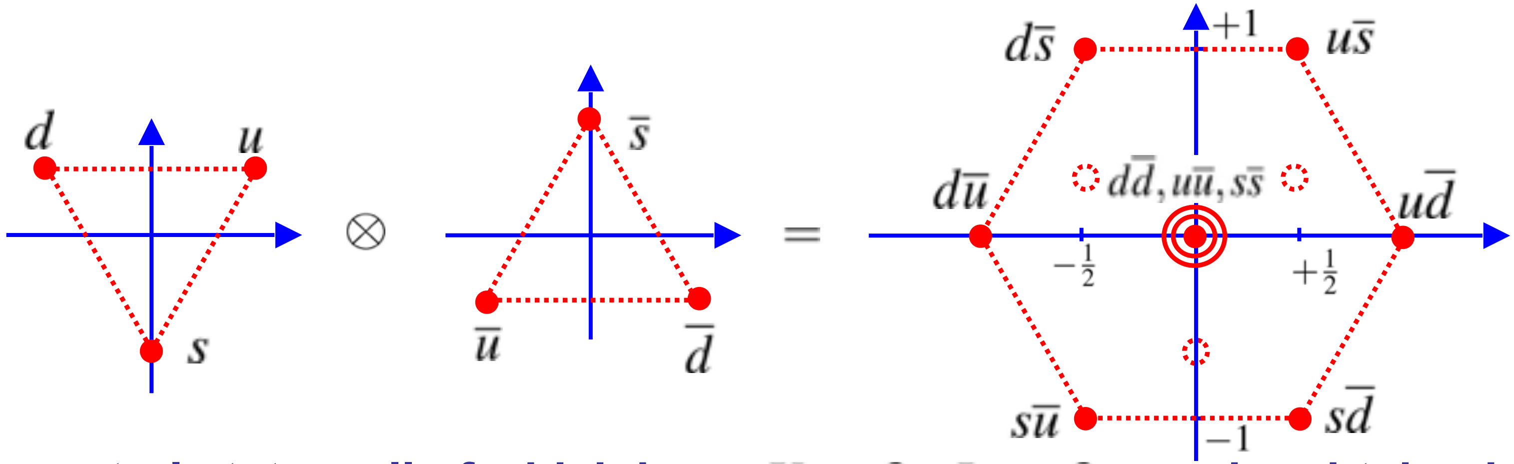
$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



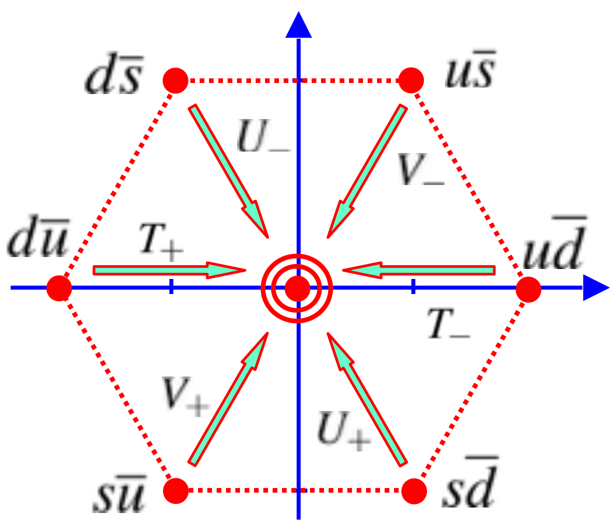


Light (uds) Mesons

- Use ladder operators to construct **uds** mesons from the nine possible $q\bar{q}$ states



- The three central states, all of which have $Y = 0;\ I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$

$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

- Only **two** of these six states are linearly independent.
- But there are **three** states with $Y = 0;\ I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \qquad |u\bar{u}\rangle - |s\bar{s}\rangle \qquad |d\bar{d}\rangle - |s\bar{s}\rangle$$

★ If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- **Experimentally** observe three light mesons with m~140 MeV: $\pi^+,\ \pi^0,\ \pi^-$

- Identify one state (the π^0) with the isospin triplet (derived previously)

$$\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the π^0

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1|\psi_2\rangle = 0; \quad \langle\psi_2|\psi_2\rangle = 1$

$$\rightarrow \quad \psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to ψ_1 and ψ_2

$$\rightarrow \quad \psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

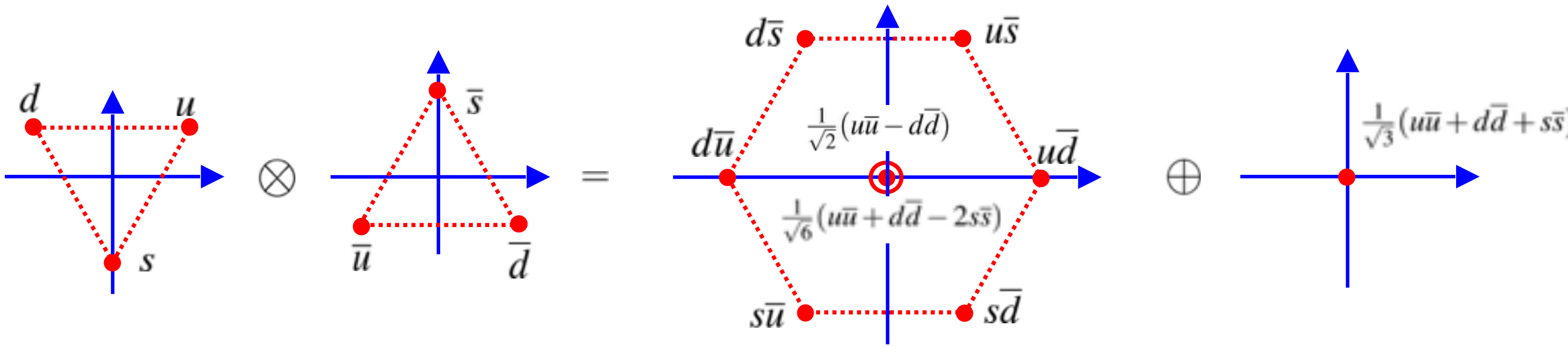
SINGLET

★ It is easy to check that ψ_3 is a singlet state using ladder operators

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a "flavourless" singlet

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET



- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$

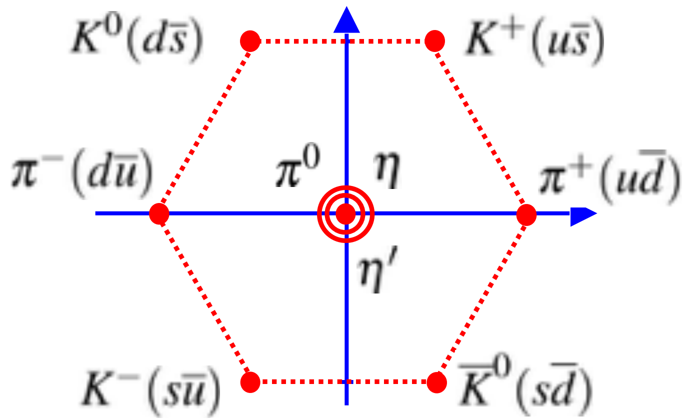
★ Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

TRIPLET of spin-1 states: $|1,-1\rangle,\ |1,0\rangle,\ |1,+1\rangle$

spin-0 SINGLET: $|0,0\rangle$

- These spin triplet states are connected by ladder operators just as the meson *uds* octet states are connected by SU(3) flavour ladder operators
- The singlet state carries no angular momentum – in this sense the SU(3) flavour singlet is "flavourless"

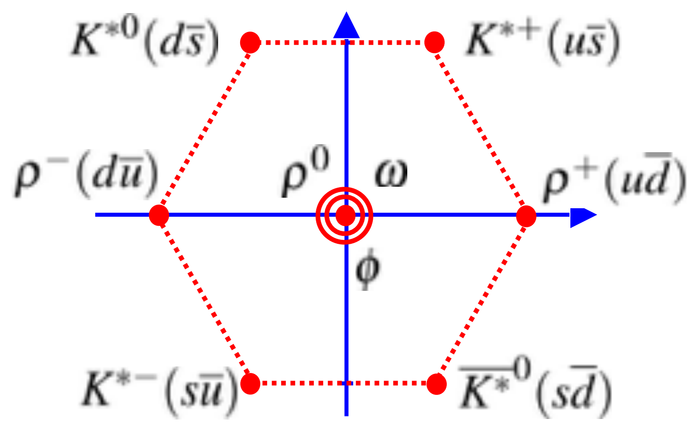
PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)



- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states. Empirically find:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \leftarrow \text{singlet}$$

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately "ideally mixed":

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

MASSES

$\pi^\pm$: 140 MeV	π^0 : 135 MeV
$K^\pm$: 494 MeV	$K^0/\bar{K}^0$: 498 MeV
η : 549 MeV	η' : 958 MeV

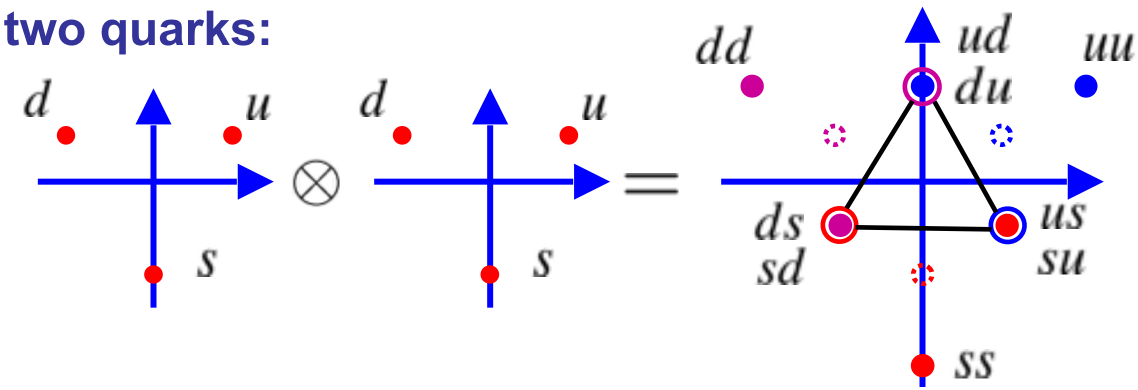
$\rho^\pm$: 770 MeV	ρ^0 : 770 MeV
$K^{*\pm}$: 892 MeV	$K^{*0}/\overline{K^*}^0$: 896 MeV
ω : 782 MeV	ϕ : 1020 MeV

Combining uds Quarks to form Baryons

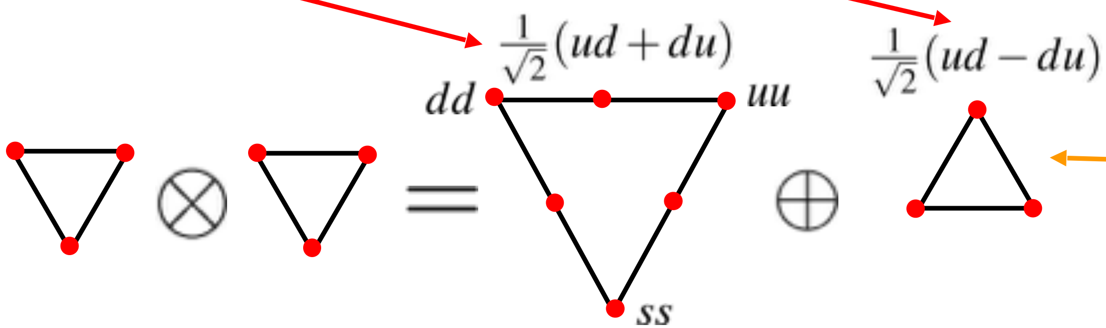
★ **Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.**

★ **<u>Everything we do here is relevant to the treatment of colour</u>**

- **First combine two quarks:**



★**Yields a <u>symmetric</u> sextet and <u>anti-symmetric</u> triplet:** $3 \otimes 3 = 6 \oplus \overline{3}$

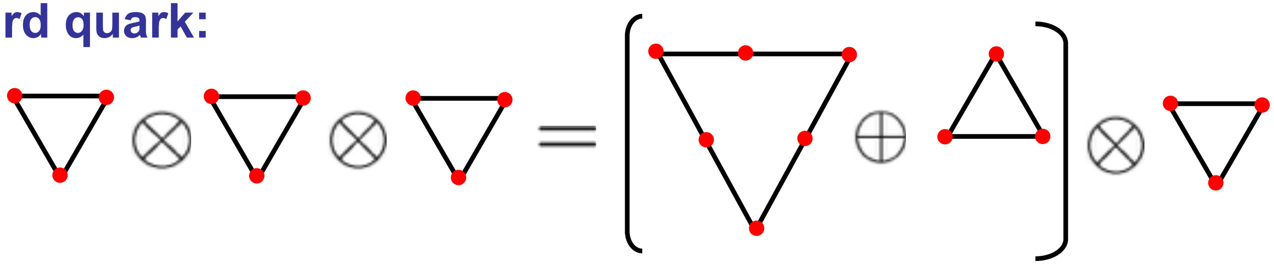


Same "pattern" as the anti-quark representation

SYMMETRIC

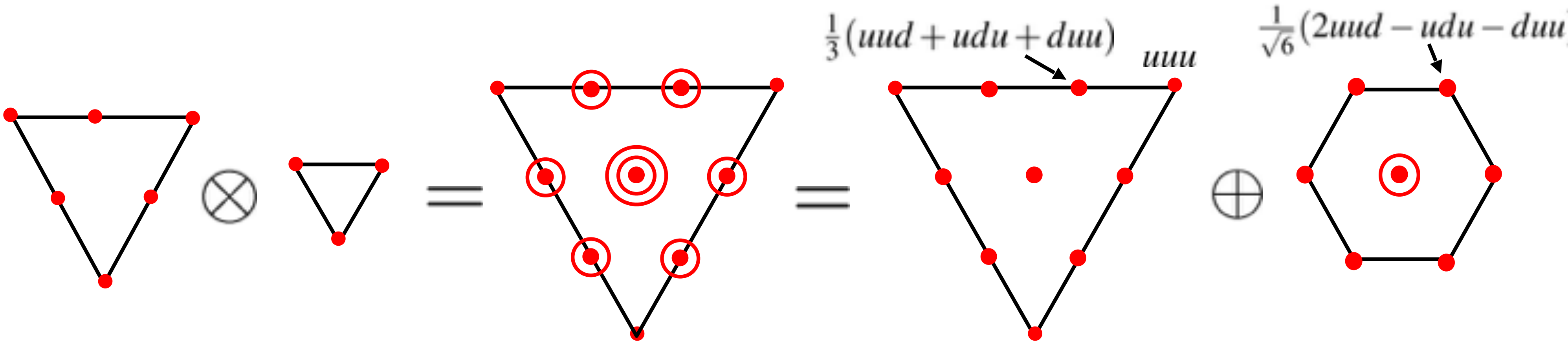
ANTI-SYMMETRIC

- Now add the third quark:



- Best considered in two parts, building on the **sextet** and **triplet**. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

❶ **Building on the sextet:** $3 \otimes 6 = 10 \oplus 8$

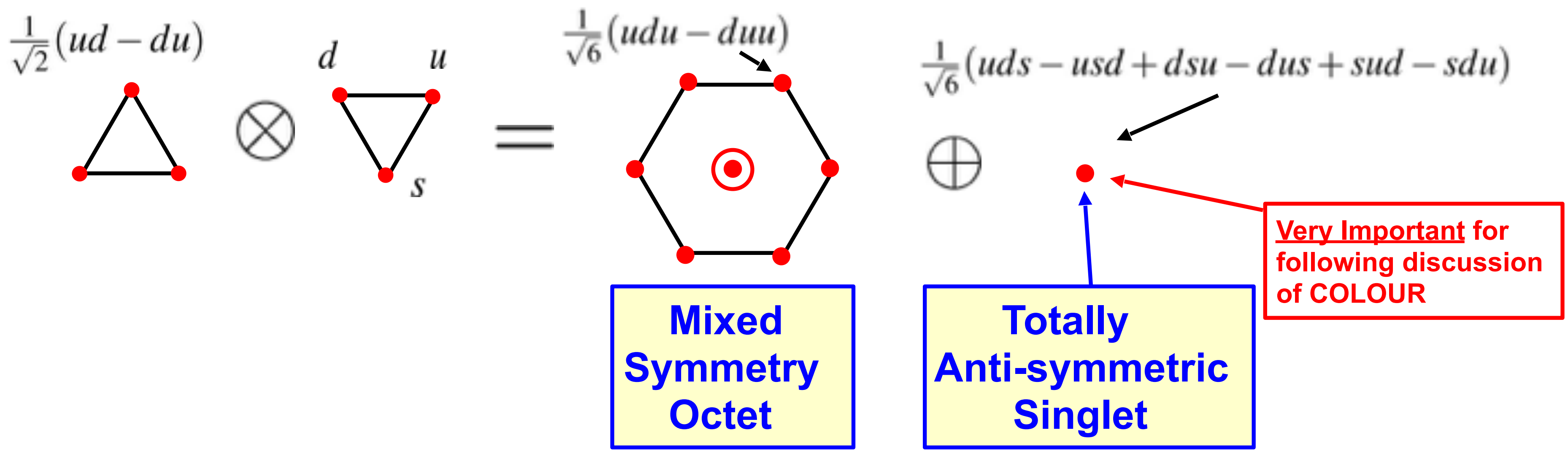


Symmetric Decuplet

Mixed Symmetry Octet

❷ Building on the triplet:

- Just as in the case of uds mesons we are combining $\bar{3} \times 3$ and again obtain an octet and a singlet



- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

★ In summary, the combination of three ***uds*** quarks decomposes into

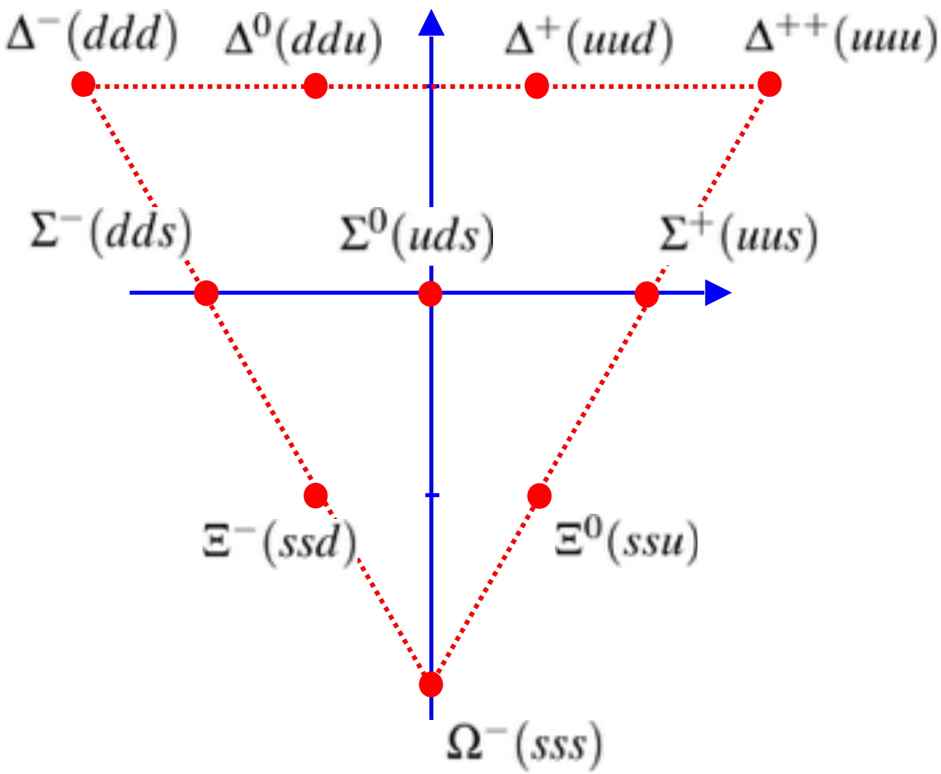
$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

Baryon Decuplet

★ The baryon states (L=0) are:

- the **spin 3/2 decuplet** of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$

BARYON DECUPLET (L=0, S=3/2, J=3/2, P= +1)



Mass in MeV

$\Delta(1232)$

$\Sigma(1318)$

$\Xi(1384)$

$\Omega(1672)$

★ If SU(3) flavour were an exact symmetry all masses would be the same (broken symmetry)

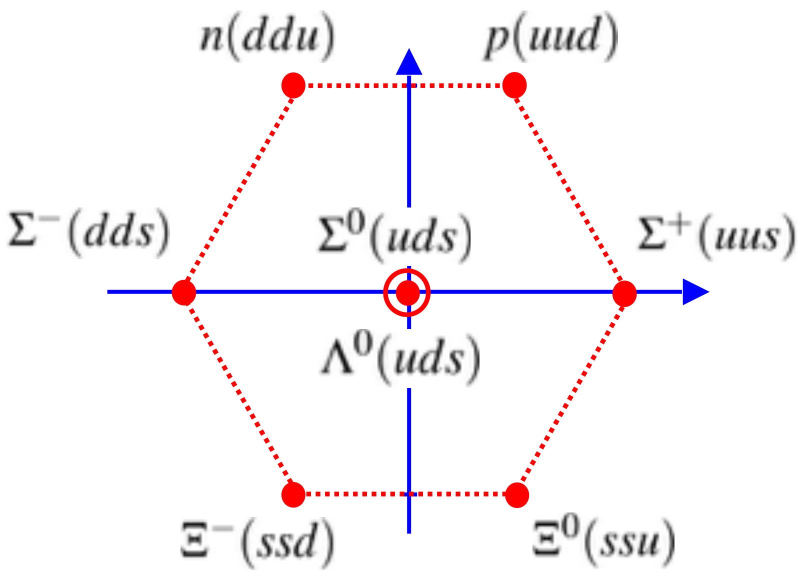
Baryon Octet

★ The **spin 1/2 octet** is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S)+\beta\phi(M_A)\chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

BARYON OCTET (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

939

Σ(1193)
Λ(1116)

Ξ(1318)

★ **NOTE:** Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

Summary

★ Considered SU(2) **ud** and SU(3) **uds** flavour symmetries

★ Although these flavour symmetries are only approximate can still be used to explain observed multiplet structure for mesons/baryons

★ In case of SU(3) flavour symmetry results, e.g. predicted wave-functions should be treated with a pinch of salt as $m_s \neq m_{u/d}$

★ Introduced idea of singlet states being “spinless” or “flavourless”

★ In the next handout apply these ideas to colour and QCD…

Appendix: the SU(2) anti-quark representation

Non-examinable

- Define anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$
- The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $q' = Uq$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow{\text{Complex conjugate}} \begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

- Express in terms of anti-quark doublet

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}' = U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}$$

- Hence $\bar{q}$ transforms as

$$\bar{q}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}$$

- In general a 2x2 unitary matrix can be written as

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

- Giving

$$\begin{aligned} \overline{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \\ &= U\overline{q} \end{aligned}$$

- Therefore the anti-quark doublet $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$

  transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$

★ **NOTE:** this is a special property of SU(2) and for SU(3) there is no analogous representation of the anti-quarks