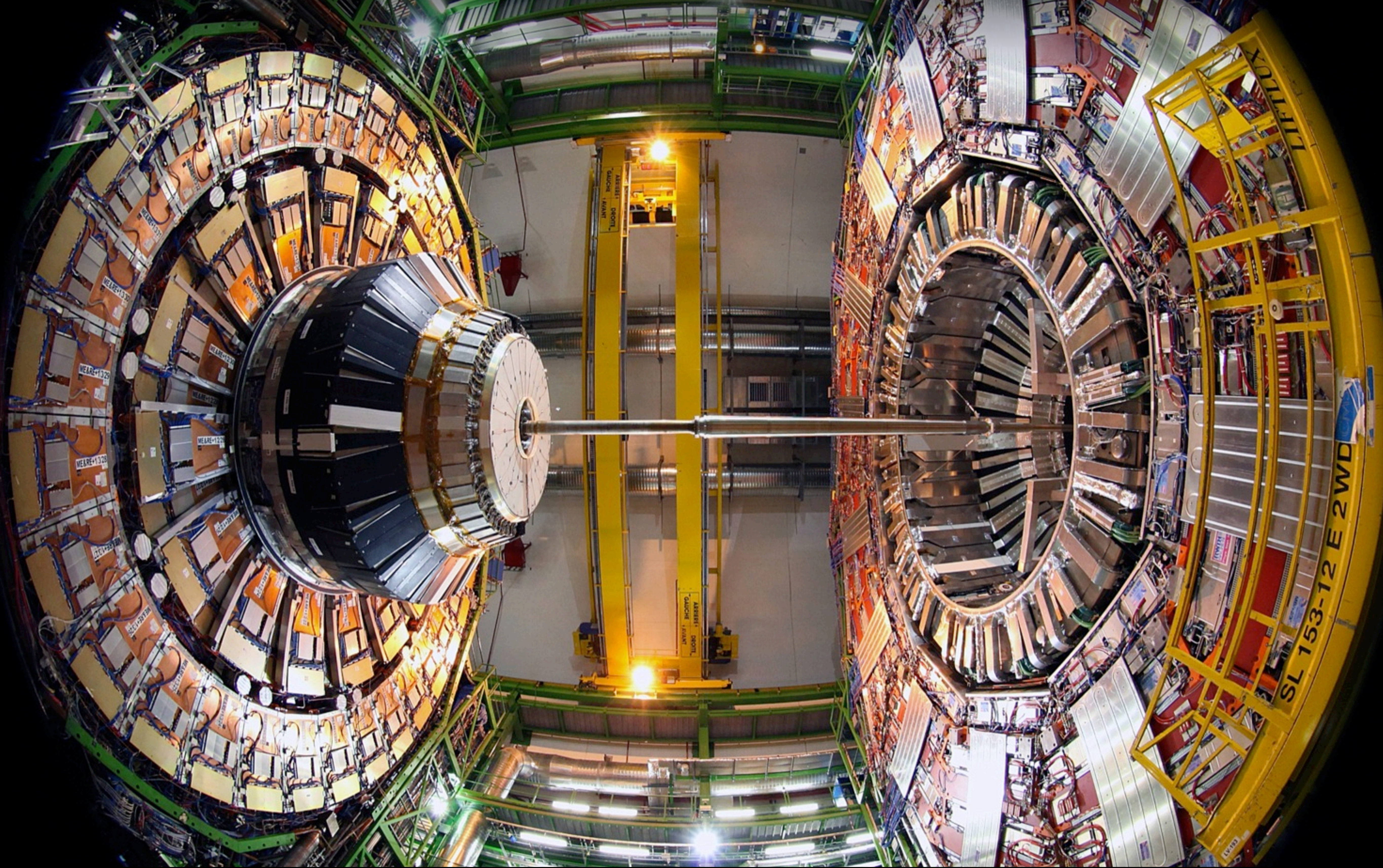


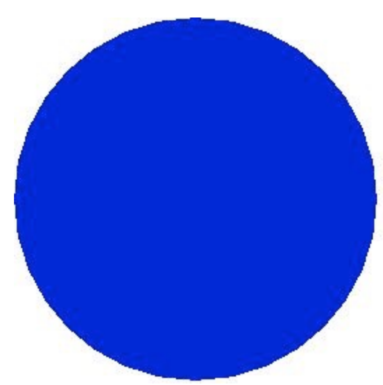
散射实验（量子世界的显微镜）

能量和空间尺度

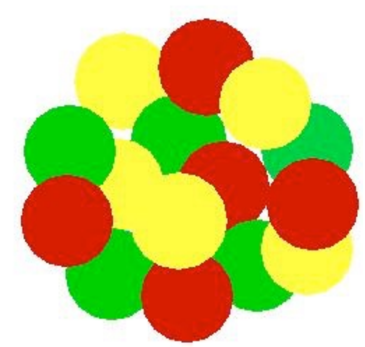
加速器：　强力的“显微镜”

高能加速的粒子束，帮助我们看清细微的结构

$$E \sim \frac{1}{x}$$



低能量粒子束



高能量粒子束

投石问路：高能散射实验

固定靶实验

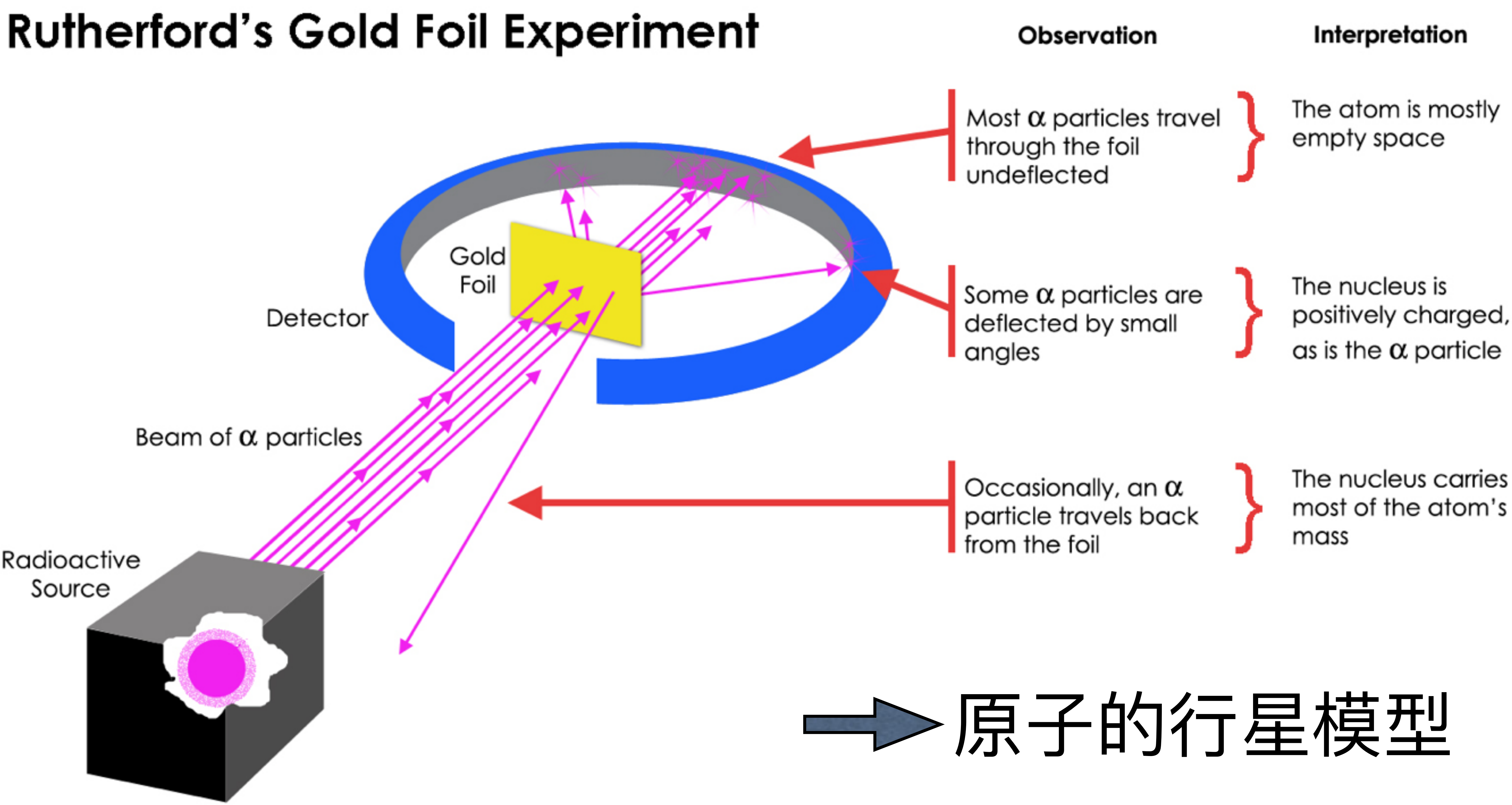
$$E_{\rm cm} \propto \sqrt{E_{\rm in}}$$

对撞机实验

$$E_{\rm cm} \propto E_{\rm in}$$

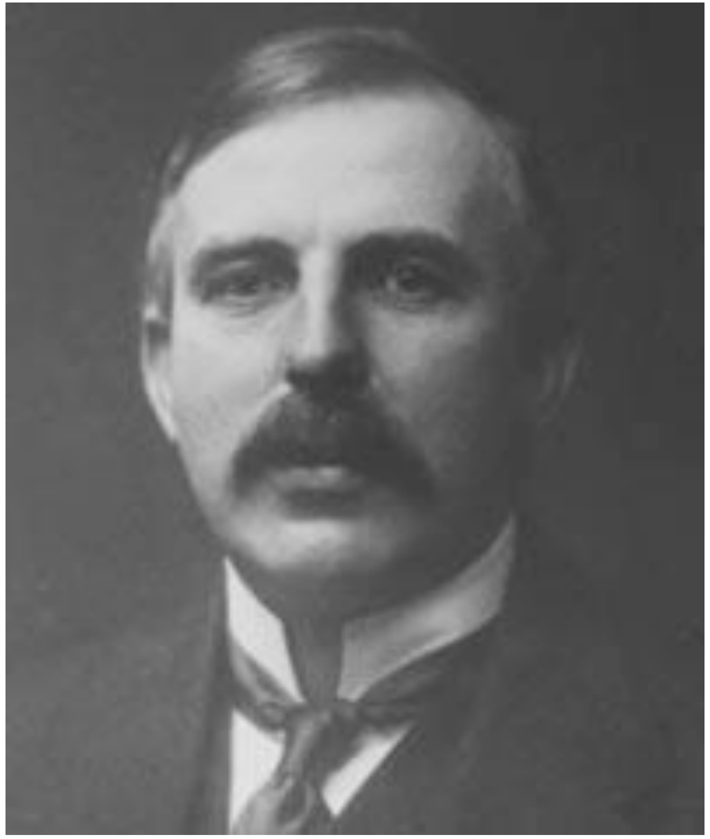
卢瑟福散射实验

对撞实验鼻祖

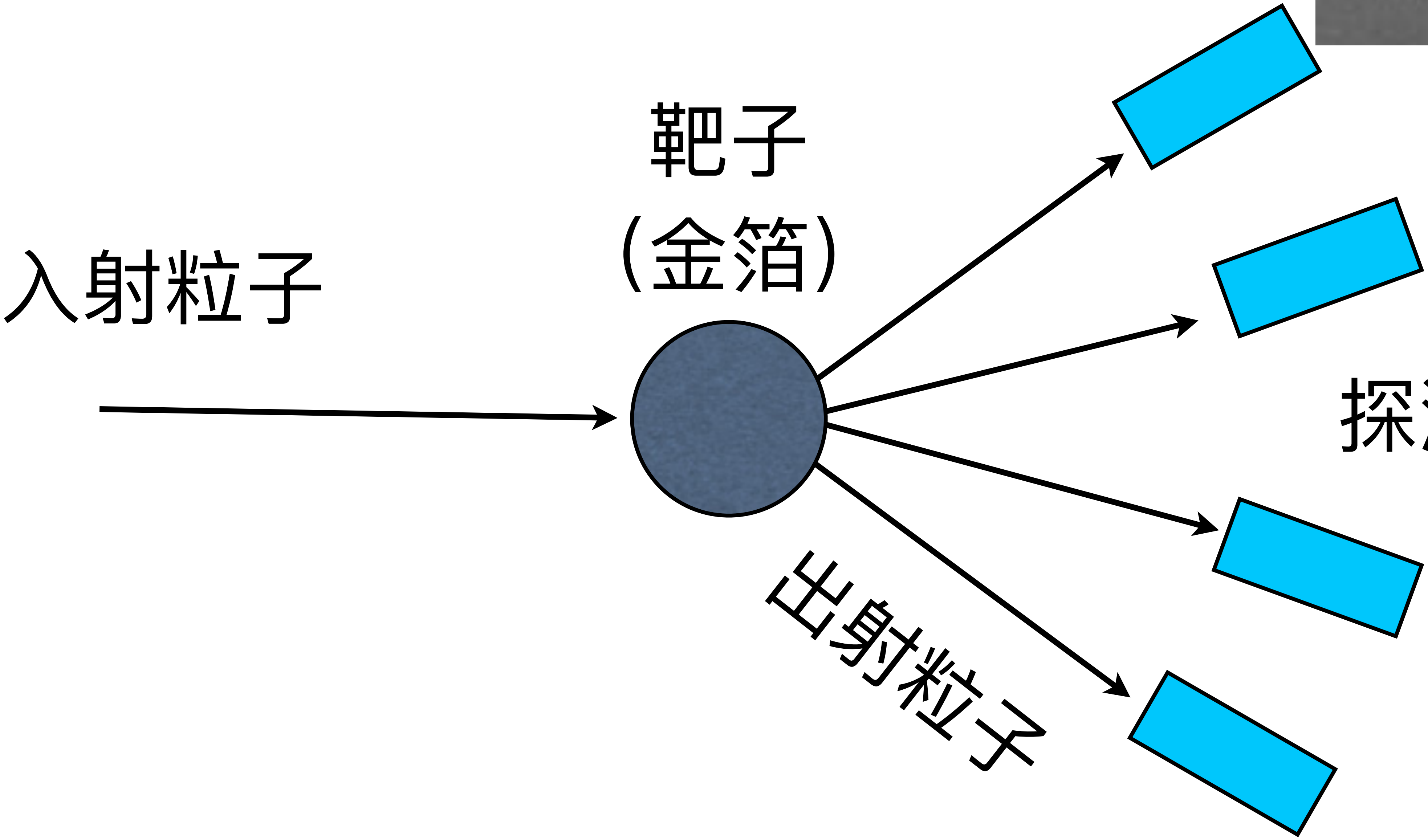


原子的行星模型

卢瑟福散射实验



对撞实验鼻祖



散射相关的物理量

入射粒子的亮度：单位时间内通过单位面积的粒子数

$$j_{\rm inc} = \frac{dN_{\rm inc}}{dA\ dt}$$

散射粒子：单位时间内在(θ, ϕ)方向的小立体角内探测到的粒子数 $\quad \dfrac{dN_{\rm sc}}{d\Omega\ dt}$

微分散射截面

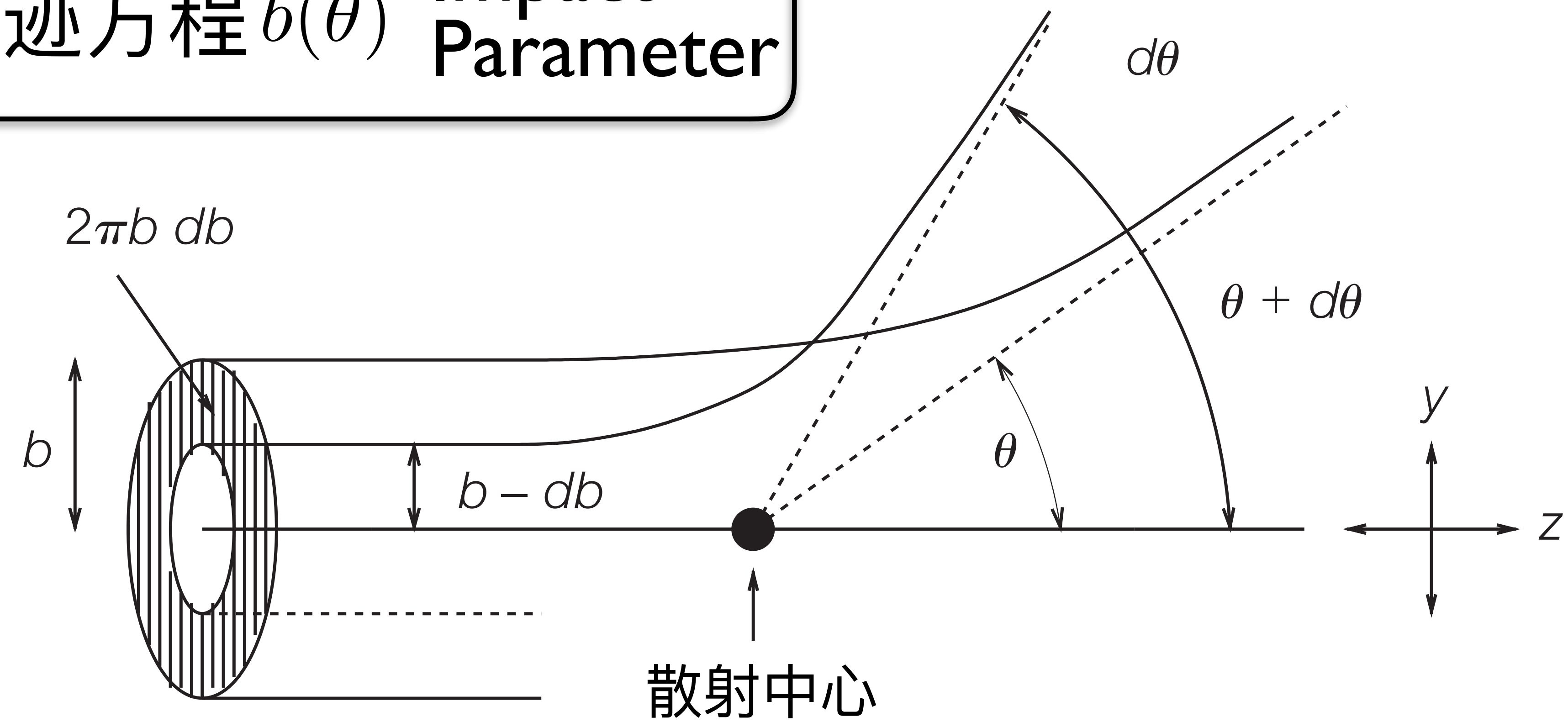
$$\frac{d\sigma}{d\Omega}(\theta, \phi) = \frac{\frac{dN_{\rm sc}}{d\Omega\, dt}}{\frac{dN_{\rm inc}}{dA\, dt}} \qquad \left[\frac{d\sigma}{d\Omega}\right] = [\sigma] = [L]^2$$

总散射截面 $\quad \sigma = \displaystyle\int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \frac{d\sigma}{d\Omega}(\theta, \phi)$

$$m\ddot{\vec{r}} = \vec{F} \rightarrow \text{轨迹方程}\, b(\theta)$$ Impact Parameter

初始条件(渐进态)

$$y(t=-\infty) = b$$
$$\dot{y}(t=-\infty) = 0$$
$$z(t=-\infty) = -\infty$$
$$\dot{z}(t=-\infty) = \sqrt{2mE}$$



阴影区域($d\sigma$)的入射粒子被散射到$(\theta, \theta + d\theta)$之间

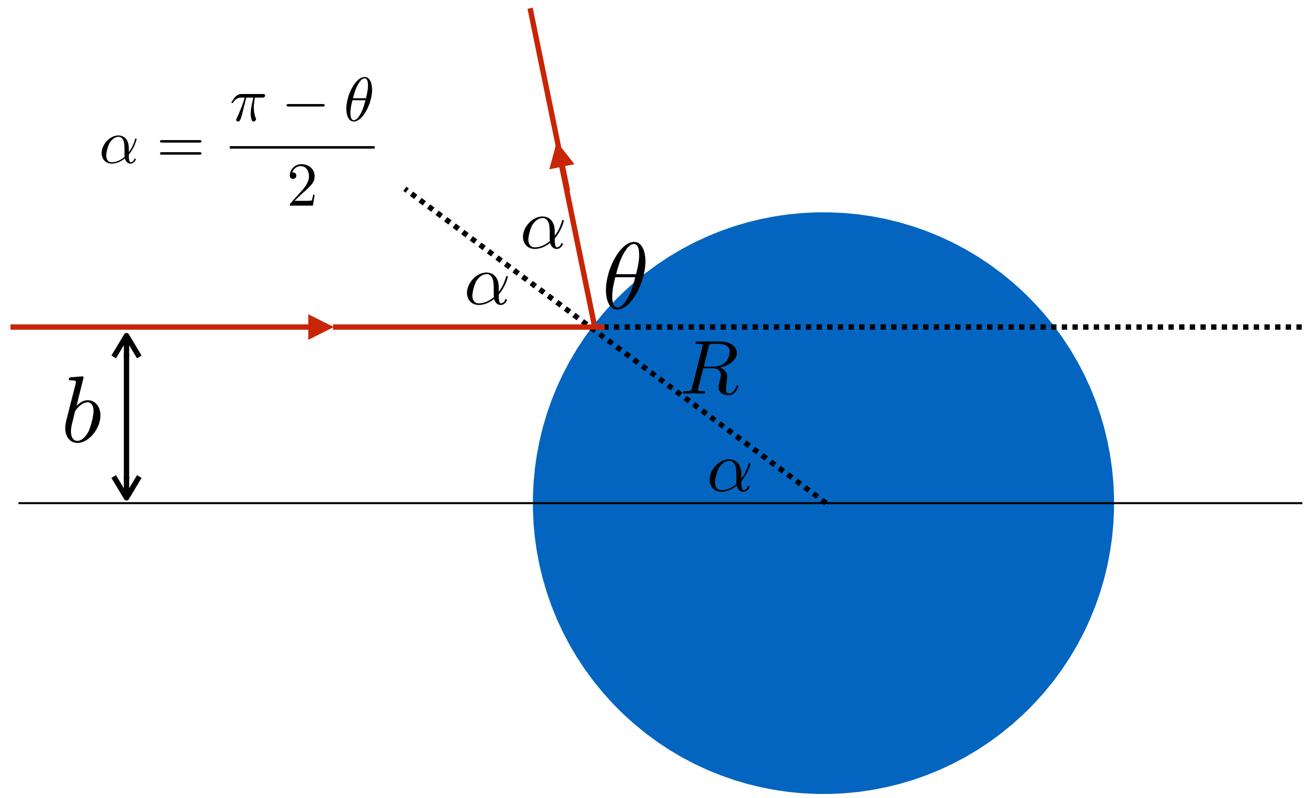
$$\frac{dN_{\rm sc}}{dt} = d\sigma \frac{dN_{\rm inc}}{dA\ dt}$$

$$d\sigma = 2\pi b db$$

$$\frac{d\sigma}{d\theta} = 2\pi b(\theta) \left| \frac{db(\theta)}{d\theta} \right|$$

$$\frac{d\sigma}{d\Omega} = \left[\frac{1}{2\pi \sin\theta}\right] 2\pi b(\theta) \left| \frac{db(\theta)}{d\theta} \right| = \frac{b(\theta)}{\sin\theta} \left| \frac{db(\theta)}{d\theta} \right|$$

经典刚球散射



$$b = R\sin\alpha = R\cos\frac{\theta}{2}$$

$$\rightarrow \left|\frac{db}{d\theta}\right| = \frac{R}{2}\sin\frac{\theta}{2}$$

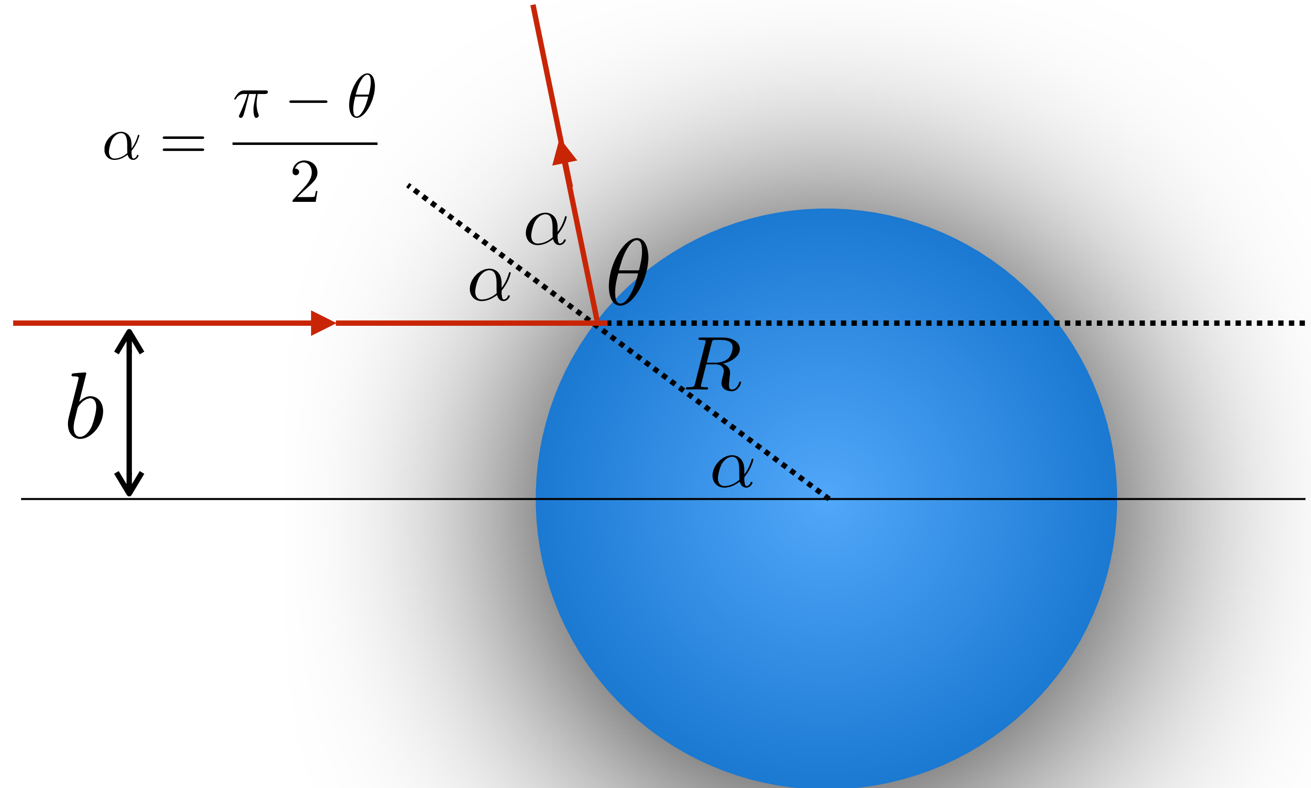
$$\frac{d\sigma}{d\theta} = 2\pi b(\theta)\left|\frac{db}{d\theta}\right| = \pi R^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\pi R^2}{2}\sin\theta$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{d\sigma}{\sin\theta d\theta d\phi} = \frac{d\sigma}{d\theta}\frac{1}{2\pi\sin\theta} = \frac{R^2}{4}$$ 微分散射截面

$$\rightarrow \sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{R^2}{4}(4\pi) = \pi R^2$$ 总散射截面

库仑势散射

$$V(r) = \frac{A}{r}$$



$$b(\theta) = \frac{A}{2E} \cot \frac{\theta}{2}$$

$$\rightarrow \quad \frac{db}{d\theta} = -\frac{A/2E}{2\sin^2 \frac{\theta}{2}}$$

$$\frac{d\sigma}{d\theta} = 2\pi b(\theta) \left| \frac{db}{d\theta} \right| = \pi \left(\frac{A}{2E} \right)^2 \frac{\cos(\theta/2)}{\sin^3(\theta/2)}$$

$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{d\sigma}{\sin\theta d\theta d\phi} = \frac{A^2}{16E^2} \frac{1}{\sin^4(\theta/2)}$$ 微分散射截面

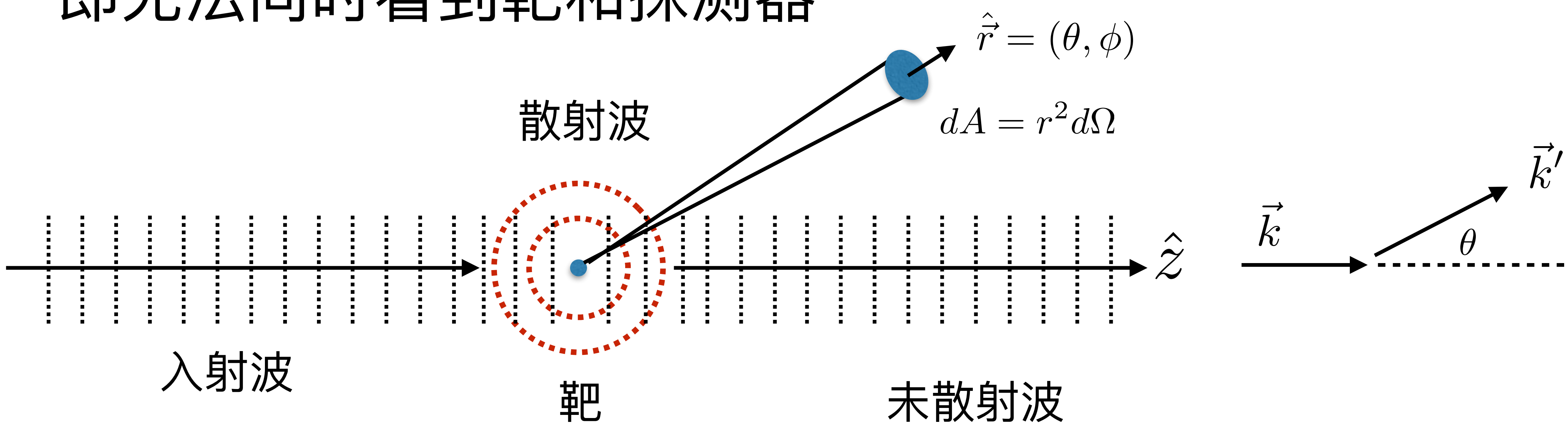
$$\rightarrow \quad \sigma = \int d\Omega \frac{d\sigma}{d\Omega} \to \infty$$ （源于里程无穷大） 总散射截面

量子散射实验

要求：

1）波包具有较大的空间尺寸，散射过程中扩散效应不显著；

2）波包大于靶的特征尺度，但小于实验装置的尺度，
即无法同时看到靶和探测器



入射粒子可以用平面波近似

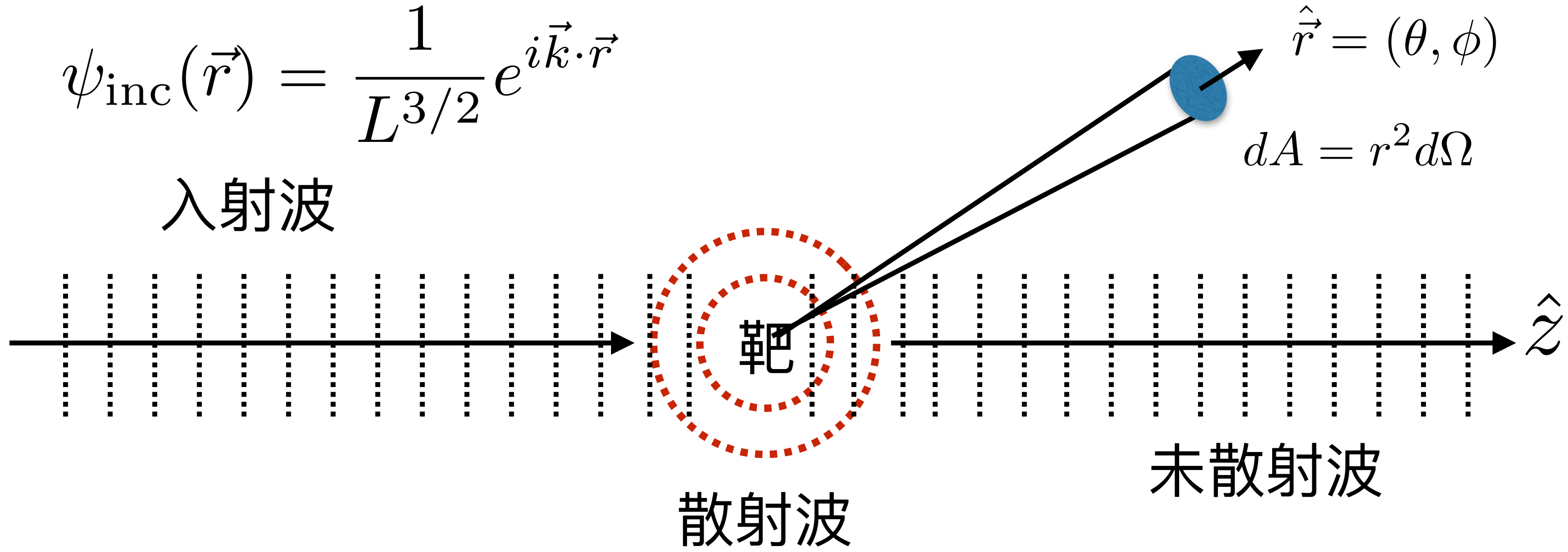
我们仅考虑弹性散射

宏观和微观上的差别：

只要入射波的波包大小比靶尺度大几倍或十几倍，
我们就可以将入射粒子视作平面波

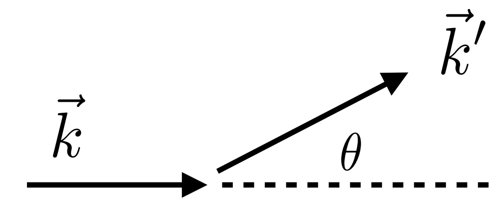
弹性散射过程的波函数

$$\psi_{\text{inc}}(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$



弹性散射

$k = k'$



$$\vec{k}\cdot\vec{r} = kz = kr\cos\theta$$

$$\psi_{\text{sc}}(r) = \frac{1}{L^{3/2}} f(\hat{n} = \hat{\vec{r}}) \frac{e^{ikr}}{r} = \frac{1}{L^{3/2}} \underbrace{f(\theta, \phi)}_{\text{散射振幅}} \frac{e^{ikr}}{r}$$

总波函数

$$\psi(\vec{r}) = \psi_{\text{inc}}(\vec{r}) + \psi_{\text{sc}}(\vec{r}) = \frac{1}{L^{3/2}} \left(e^{i\vec{k}\cdot\vec{r}} + f(\theta, \phi) \frac{e^{ikr}}{r} \right)$$

$$\sim e^{ikr(1-\cos\theta)}$$

干涉项的相因子

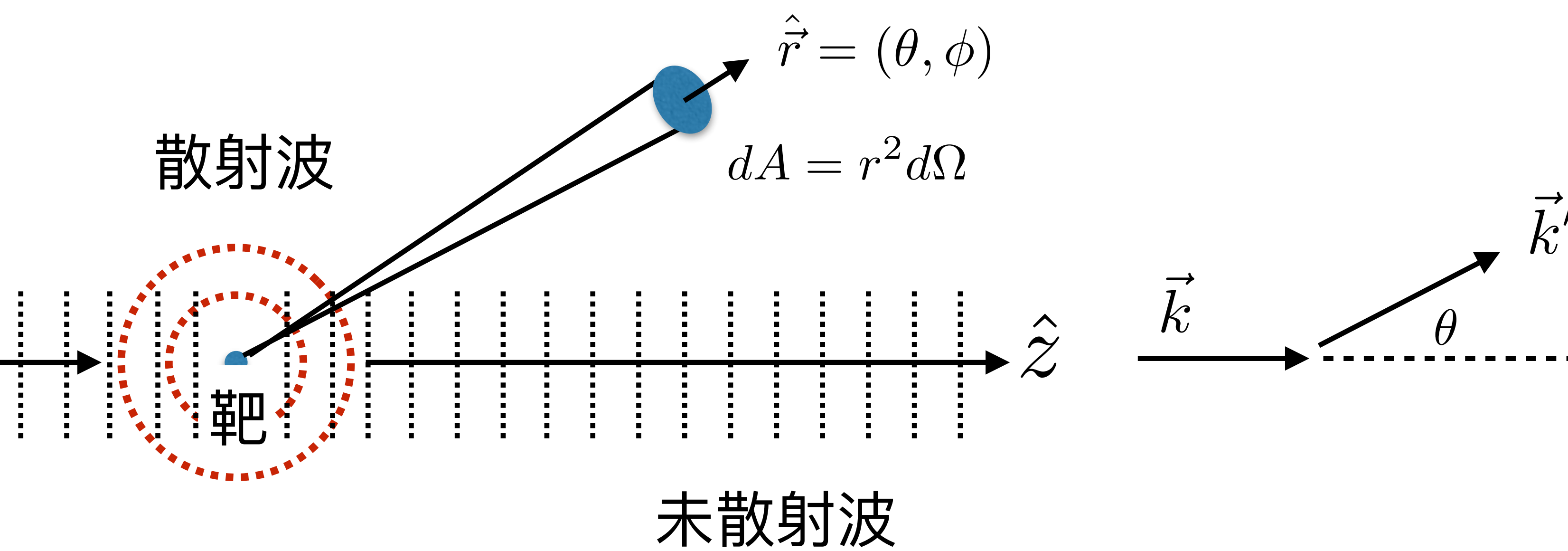
$kr \gg 1$ 相位因子振荡剧烈相消

$\theta \to 0$ 无干涉相消，但在Z轴方向无探测器

几率流通矢

$$\vec{j}(\vec{r}) = -i\frac{\hbar}{2m}\Big(\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r}\nabla\psi^*(\vec{r}))\Big)$$

$$\psi_{\rm inc}(\vec{r}) = \frac{1}{L^{3/2}}e^{i\vec{k}\cdot\vec{r}}$$



$$\nabla\psi_{\rm inc}(\vec{r}) = \frac{1}{L^{3/2}}i\vec{k}e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{j}_{\rm inc} = \frac{1}{L^3}\frac{\hbar\vec{k}}{m} = \frac{1}{L^3}\vec{v}$$

单位时间内通过
单位面积的粒子数 $\longrightarrow$ $\vec{j}_{\rm inc} = \dfrac{dN_{\rm inc}}{dAdt}$

散射波的几率流通矢

$$\psi_{\rm sc}(r) = \frac{1}{L^{3/2}} f(\hat{n} = \hat{\vec{r}}) \frac{e^{ikr}}{r} = \frac{1}{L^{3/2}} \underbrace{f(\theta,\phi)}_{\text{散射振幅}} \frac{e^{ikr}}{r}$$

$$\nabla\psi_{\rm sc}(\vec{r}) = \frac{1}{L^{3/2}} \left[\vec{e}_r f(\theta,\phi) \left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) + \vec{e}_\theta \frac{1}{r} \frac{\partial f(\theta,\phi)}{\partial\theta} \frac{e^{ikr}}{r} + \vec{e}_\phi \frac{1}{r\sin\theta} \frac{\partial f(\theta,\phi)}{\partial\phi} \frac{e^{ikr}}{r} \right]$$

$$\vec{j}_{\rm sc}(\vec{r}) = \vec{e}_r \frac{1}{L^3} \frac{\hbar k}{m} \frac{|f(\theta,\phi)|^2}{r^2} + \mathcal{O}(\frac{1}{r^3})$$

任何$1/r^3$或更高阶项
在远处贡献可忽略

单位时间内散射到(θ,ϕ)方向
小面积内的粒子数

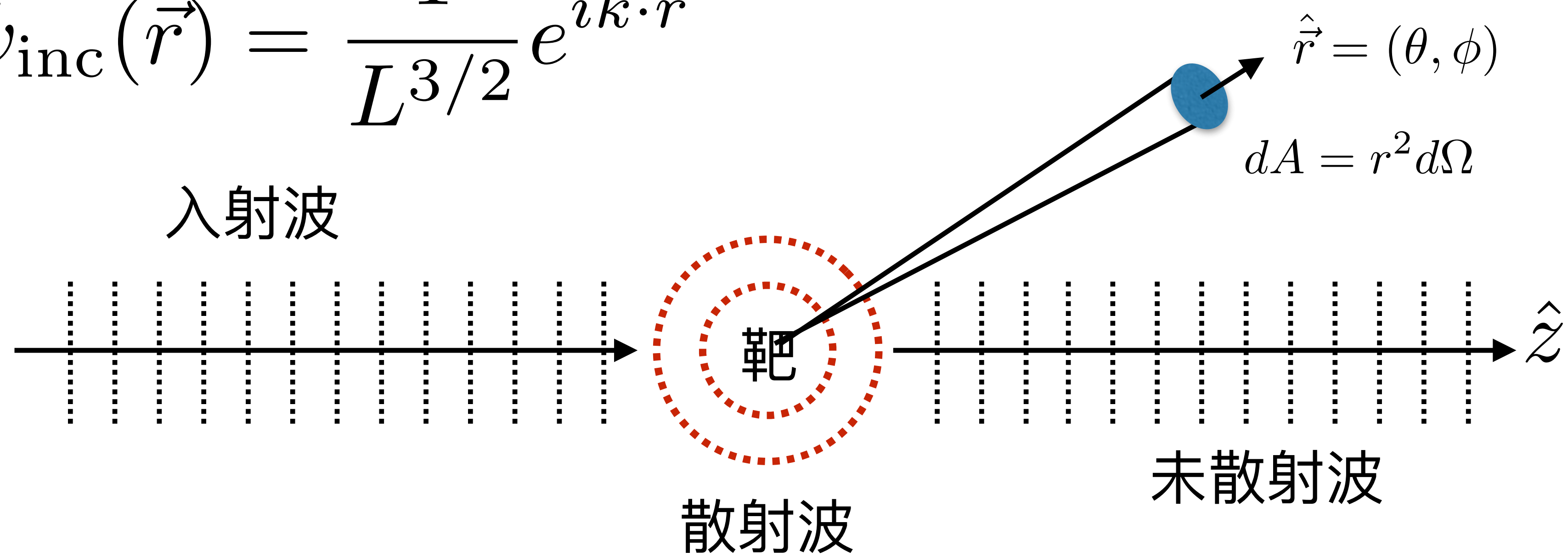
$$\vec{j}_{\rm sc}(\vec{r}) d\vec{A} = \left[\vec{e}_r \frac{1}{L^3} \frac{\hbar k}{m} \frac{|f(\theta,\phi)|^2}{r^2} \right] \vec{e}_r r^2 d\Omega$$
$$= \frac{1}{L^3} \frac{\hbar k}{m} |f(\theta,\phi)|^2 d\Omega$$

$\Rightarrow$

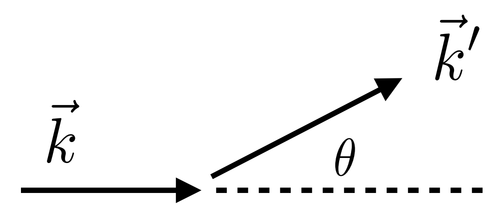
$$\frac{dN_{\rm sc}}{d\Omega dt} = \frac{\vec{j}_{\rm sc} \cdot d\vec{A}}{d\Omega}$$
$$= \frac{1}{L^3} \frac{\hbar k}{m} |f(\theta,\phi)|^2$$

微分散射截面

$$\psi_{\rm inc}(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$



弾性散射
$k = k'$



$$\vec{k}\cdot\vec{r} = kz = kr\cos\theta$$

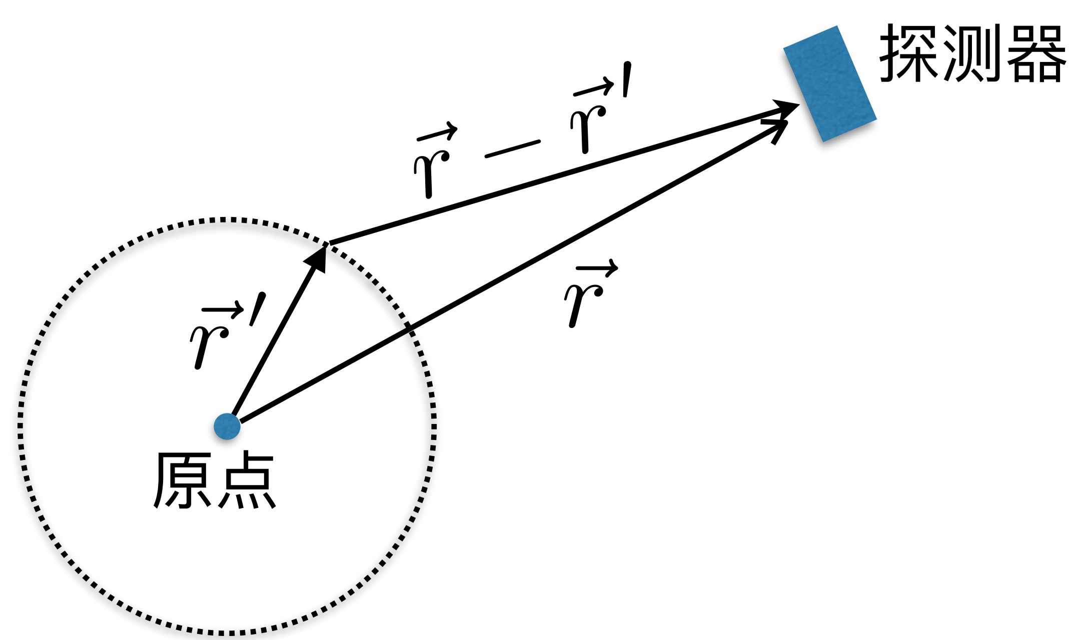
$$\psi_{\rm sc}(r) = \frac{1}{L^{3/2}} f(\hat{n} = \hat{\vec{r}}) \frac{e^{ikr}}{r} = \frac{1}{L^{3/2}} \underbrace{f(\theta, \phi)}_{\text{散射振幅}} \frac{e^{ikr}}{r}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\dfrac{dN_{\rm sc}}{d\Omega dt}}{\dfrac{dN_{\rm inc}}{dA\ dt}} = \frac{\dfrac{1}{L^3}\dfrac{\hbar k}{m}|f(\theta,\phi)|^2}{\dfrac{1}{L^3}\dfrac{\hbar k}{m}} = |f(\theta,\phi)|^2}$$

散射的波动方程和近似

散射的波动方程

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})$$



$$k^2 = \frac{2mE}{\hbar^2}$$

$$\left(\nabla^2 + k^2\right)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r})$$

积分方程形式解 Lippman-Schwinger方程

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$$

$$= e^{i\vec{k}\cdot\vec{r}} + \int \underbrace{\left(-\frac{m}{2\pi\hbar^2}\right)\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}}_{G(\vec{r},\vec{r}\,')\ \text{格林函数}} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$$

验证 $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \left(-\frac{m}{2\pi\hbar^2}\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$

满足微分形式的波动方程

$$\left(\nabla^2 + k^2\right)\psi(\vec{r}) = \frac{2m}{\hbar^2} V(\vec{r})\psi(\vec{r})$$

第1项 $e^{i\vec{k}\cdot\vec{r}}$ ： $\left(\nabla^2 + k^2\right) e^{i\vec{k}\cdot\vec{r}} = (-k^2 + k^2)e^{i\vec{k}\cdot\vec{r}} = 0$

第2项需要计算 $\left(\nabla^2 + k^2\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$

$$\nabla^2 \frac{e^{ikr}}{r} = -k^2 \frac{e^{ikr}}{r} + e^{ikr}\nabla^2 \frac{1}{r} \qquad\qquad \nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$$

$$\begin{aligned}\left(\nabla^2 + k^2\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} &= (-k^2 + k^2)\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} + e^{ik|\vec{r}-\vec{r}\,'|}\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} \\ &= -4\pi e^{ik|\vec{r}-\vec{r}\,'|}\delta(\vec{r}-\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')\end{aligned}$$

验证 $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \left(-\frac{m}{2\pi\hbar^2}\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$

满足微分形式的波动方程

$$\left(\nabla^2 + k^2\right)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r})$$

第1项：$\left(\nabla^2 + k^2\right) e^{i\vec{k}\cdot\vec{r}} = (-k^2 + k^2)e^{i\vec{k}\cdot\vec{r}} = 0$

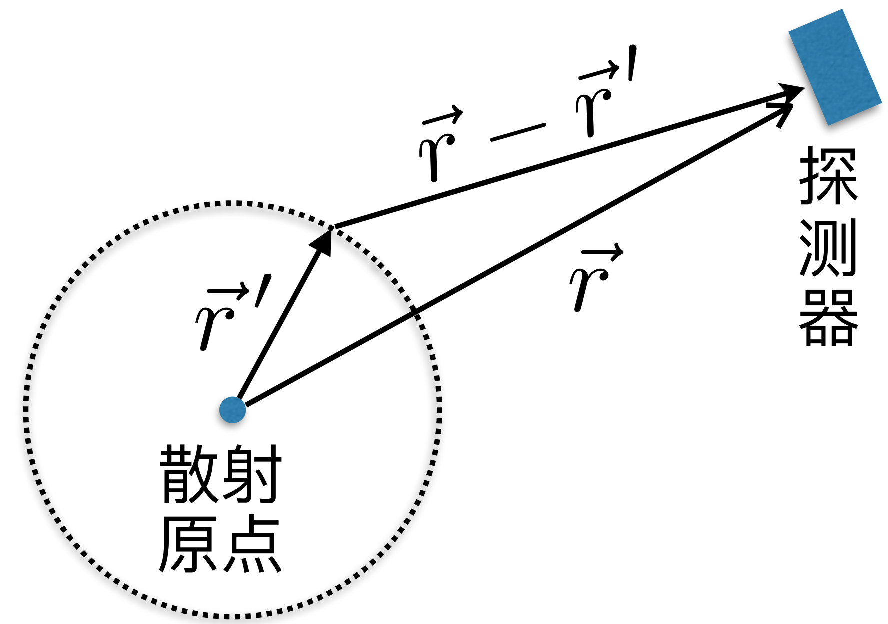
第2项：$\left(\nabla^2 + k^2\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta(\vec{r}-\vec{r}\,')$

故而

$$\begin{aligned}\left(\nabla^2 + k^2\right)\psi(\vec{r}) &= -\frac{m}{2\pi\hbar^2}(-4\pi)\int \delta(\vec{r}-\vec{r}\,')V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,' \\ &= \frac{2m}{\hbar}V(\vec{r})\psi(\vec{r})\end{aligned}$$

求解 $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \left(-\frac{m}{2\pi\hbar^2}\right)\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$

考虑 $|\vec{r}| \gg |\vec{r}\,'|$



$$K|\vec{r}-\vec{r}\,'| = k\sqrt{\vec{r}^{\,2} - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^2}$$

$$\approx kr - k\frac{\vec{r}}{r}\cdot\vec{r}\,' = kr - \vec{k}\cdot\vec{r}\,'$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\frac{1}{\left|1-\frac{\vec{r}\cdot\vec{r}\,'}{\vec{r}^{\,2}}\right|} \approx \frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}\,'}{\vec{r}^{\,2}}\right) \approx \frac{1}{r}$$

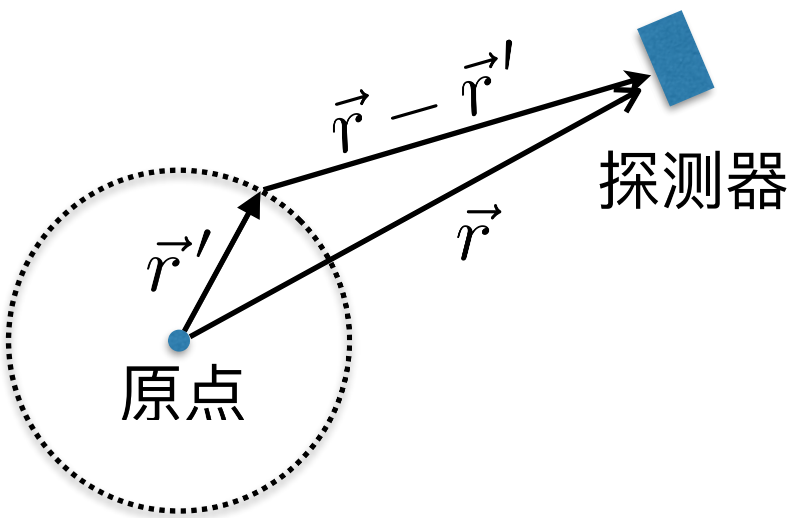
$$\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \approx \frac{e^{ikr}}{r}e^{-ik\hat{e}_r\cdot\vec{r}\,'} = \frac{e^{ikr}}{r}e^{-i\vec{k}\,'\cdot\vec{r}\,'}$$

$\vec{k}\,' \equiv k\hat{\vec{r}}$
波矢沿着探测器方向

故

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} - \left[\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'}V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'\right]\frac{e^{ikr}}{r}$$

求解 $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \left(-\frac{m}{2\pi\hbar^2}\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$



考虑 $|\vec{r}| \gg |\vec{r}\,'|$

定义 $\vec{k}\,' \equiv k\hat{\vec{r}}$ 为波矢沿着探测器方向 (θ, ϕ)

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} - \left[\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'\right]\frac{e^{ikr}}{r}$$

与 $\psi(\vec{r}) = \frac{1}{L^{3/2}}\left(e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}\right)$ 对比可知

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$$

求解 $f(\theta,\phi)$ 仍然需要知道 $\psi(\vec{r})$ 的信息 $\longrightarrow$ 近似迭代求解

近似迭代求解

1）$V(\vec{r}) = 0$ 时，$\psi^{(0)} = e^{i\vec{k}\cdot\vec{r}}$

2）代入到Lippman-Schwinger方程中

$$\psi^{(1)}(\vec{r}) = \psi^{(0)}(\vec{r}) + \int \underbrace{\left(-\frac{m}{2\pi\hbar^2}\right)\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}}_{G(\vec{r},\vec{r}\,')}\psi^{(0)}(\vec{r}\,')V(\vec{r}\,')d^3\vec{r}\,'$$

3）$\psi^{(1)}$ 代入到Lippman-Schwinger方程中

$$\Longrightarrow \psi^{(2)} \Longrightarrow \psi^{(3)} \Longrightarrow \cdots$$

$$\begin{aligned}\psi(\vec{r}) = \psi^{(0)}(\vec{r}) &+ \int d\vec{r}\,' G(\vec{r},\vec{r}\,')V(\vec{r}\,')\psi^{(0)}(\vec{r}\,') \\ &+ \int d\vec{r}\,' \int d\vec{r}\,'' G(\vec{r},\vec{r}\,')G(\vec{r}\,',\vec{r}\,'')V(\vec{r}\,'')V(\vec{r}\,') \\ &+ \cdots\end{aligned}$$

玻恩 (Born) 近似

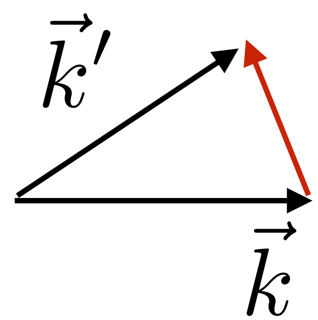
玻恩近似

势场非常微弱时，只需计算第一阶微扰修正

$$\psi^{(0)} = e^{i\vec{k}\cdot\vec{r}\,'}$$

$$f_B(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'}V(\vec{r}\,')e^{i\vec{k}\cdot\vec{r}\,'}d^3\vec{r}\,'$$

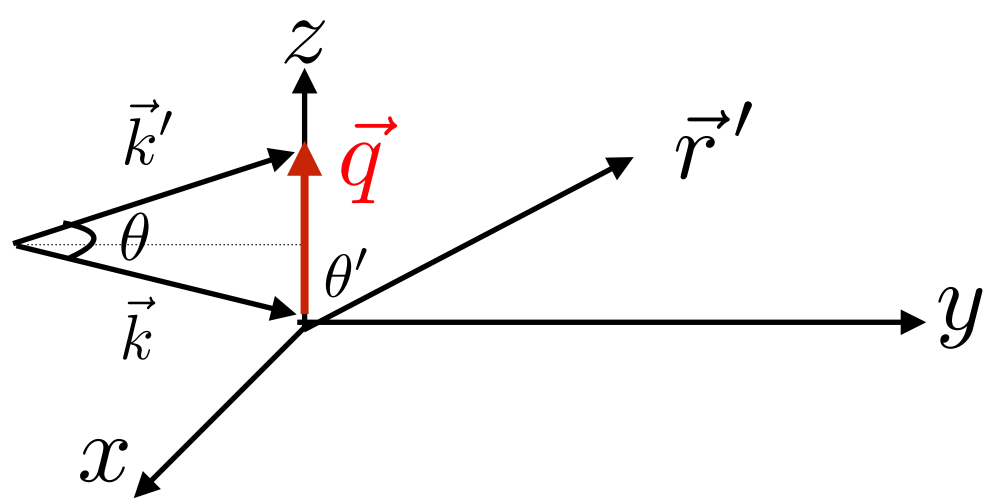
$$= -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')d\vec{r}\,'$$



$\vec{q} \equiv \vec{k}' - \vec{k}$ 散射前后粒子动量变化

散射振幅仅仅是势场 $V(\vec{r})$ 相对于 $\vec{q}$ 的傅里叶变换

对于中心势场，散射与方位角 ϕ 无关，取 $\vec{q}$ 方向为 z 轴



$$\vec{q}\cdot\vec{r}\,' = qr\,'\cos\theta'$$
$$q = |\vec{k}' - \vec{k}| = 2k\sin\frac{\theta}{2}$$

$$\begin{aligned}\int e^{-i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')d\vec{r}\,' &= \int_0^\infty dr'r'^2V(r')\int_0^{2\pi}d\phi\int_0^\pi d\theta'\sin\theta' e^{-iqr'\cos\theta'} \\ &= \frac{2\pi}{iq}\int_0^\infty dr'r'V(r')\left(e^{iqr'} - e^{-iqr'}\right) \\ &= \frac{4\pi}{q}\int_0^\infty dr'r'\sin(qr')V(r')\end{aligned}$$

$$\Rightarrow \quad f_B(q) = -\frac{2m}{\hbar^2 q}\int_0^\infty dr'r'\sin(qr')V(r')$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty dr'r'\sin(qr')V(r')\right|^2$$

玻恩近似的跃迁图像

$V(\vec{r})$ 很弱时，可将之视作为微扰

散射：常微扰作用下，$|\psi_{\vec{p}}\rangle \rightarrow |\psi_{\vec{p}'}\rangle$ 跃迁

跃迁矩阵元

$$\begin{aligned} V_{\vec{p}'\vec{p}} &= \left\langle \psi_{\vec{p}'} \middle| \hat{V} \middle| \psi_{\vec{p}} \right\rangle \\ &= \int_\Omega \frac{1}{L^{3/2}} e^{-i\frac{\vec{p}'\cdot\vec{r}}{\hbar}} V(\vec{r}) \frac{1}{L^{3/2}} e^{i\frac{\vec{p}\cdot\vec{r}}{\hbar}} d^3\vec{r} \\ &= \frac{1}{L^3} \underbrace{\int_\Omega V(\vec{r}) e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{r}} d^3\vec{r}}_{V(\vec{p}'-\vec{p})} \end{aligned}$$

能量 $E_{\vec{p}'}$ 态密度函数

$$\begin{aligned} \rho(E_{\vec{p}'}) dE_{\vec{p}'} &= \frac{L^3}{(2\pi\hbar)^3} d^3\vec{p}' \\ &= \frac{L^3}{(2\pi\hbar)^3} (p')^2 \, dp' d\Omega \end{aligned}$$

末态自由粒子

$$dE_{\vec{p}'} = \frac{p'}{m} dp'$$

故

$$\rho(E_{\vec{p}'}) = \frac{L^3 m p'}{(2\pi\hbar)^3} d\Omega$$

玻恩近似的跃迁图像

$V(\vec{r})$ 很弱时，可将之视作为微扰

散射：常微扰作用下, $|\psi_{\vec{p}}\rangle \to |\psi_{\vec{p}'}\rangle$ 跃迁

由费米黄金规则知，跃迁几率为

$$W_{\vec{p}'\vec{p}} = \frac{2\pi}{\hbar}\left|V_{\vec{p}'\vec{p}}\right|^2 \rho(E_{\vec{p}'}) = \frac{2\pi}{\hbar}\left|V_{\vec{p}'\vec{p}}\right|^2 \frac{L^3 m p'}{(2\pi\hbar)^3} d\Omega$$

$$= \frac{2\pi}{\hbar}\left|V(\vec{p}' - \vec{p})\right|^2 \frac{1}{(2\pi\hbar)^3}\frac{mp'}{L^3} d\Omega$$

散射截面定义

$$W_{\vec{p}'\vec{p}} \equiv j_{\mathrm{inc}}\sigma(\theta)d\Omega = \frac{p}{mL^3}\sigma(\theta)d\Omega = \frac{p'}{mL^3}\sigma(\theta)d\Omega$$

故而有

$$\sigma(\theta) = \frac{m^2}{4\pi^2\hbar^4}\left|V(\vec{p} - \vec{p}')\right|^2 = \frac{m^2}{4\pi^2\hbar^4}\left|\int e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{r}'} V(\vec{r}')d^3\vec{r}'\right|^2$$

$$= \frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty dr' r' \sin(qr') V(r')\right|^2$$

玻恩近似的适用条件

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \left(-\frac{m}{2\pi\hbar^2}\right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')d^3\vec{r}\,'$$

第二项贡献远小于第一项（入射波）：

$$\left| \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,') e^{i\vec{k}\cdot\vec{r}\,'} d^3\vec{r}\,' \right| \ll |\psi_{\text{inc}}(\vec{r})| = 1$$

考虑弹性散射，并且假设势函数在r=0处最大，

$$\left| \frac{m}{\hbar^2} \int_0^\infty r' e^{ikr'} V(r') dr' \int_0^\pi e^{ikr'\cos\theta'} \sin\theta' d\theta' \right| \ll 1$$

$$\Rightarrow \quad \frac{m}{\hbar^2 k} \left| \int_0^\infty V(r') \left(e^{2iqr'} - 1 \right) dr' \right| \ll 1 \qquad E_{\text{inc}} = \frac{\hbar^2 k^2}{2m}$$

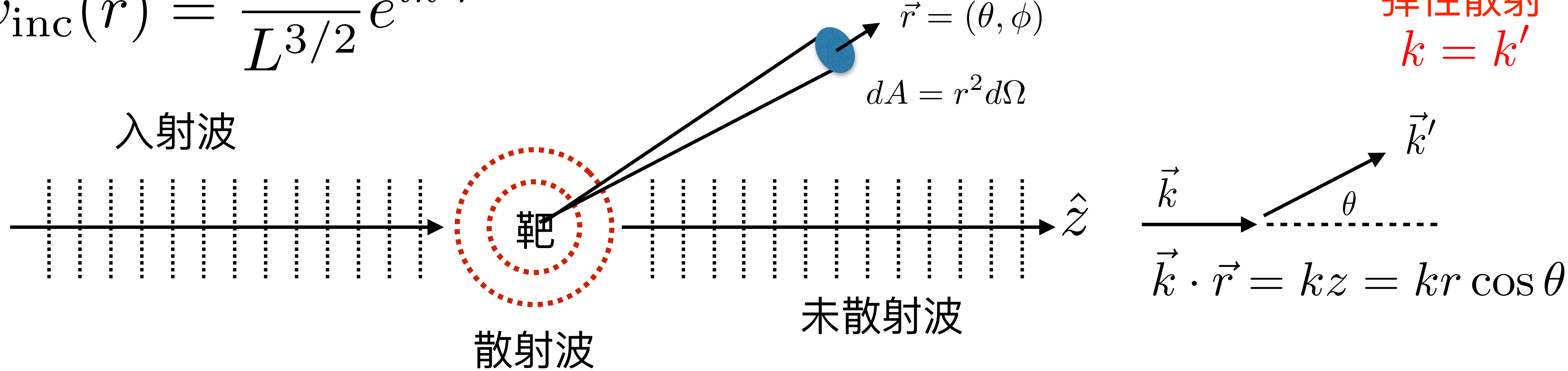
高能低势 ⟶ 平面波近似适用

分波法

不依赖于势场强弱

弹性散射的分波分析

$$\psi_{\rm inc}(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$



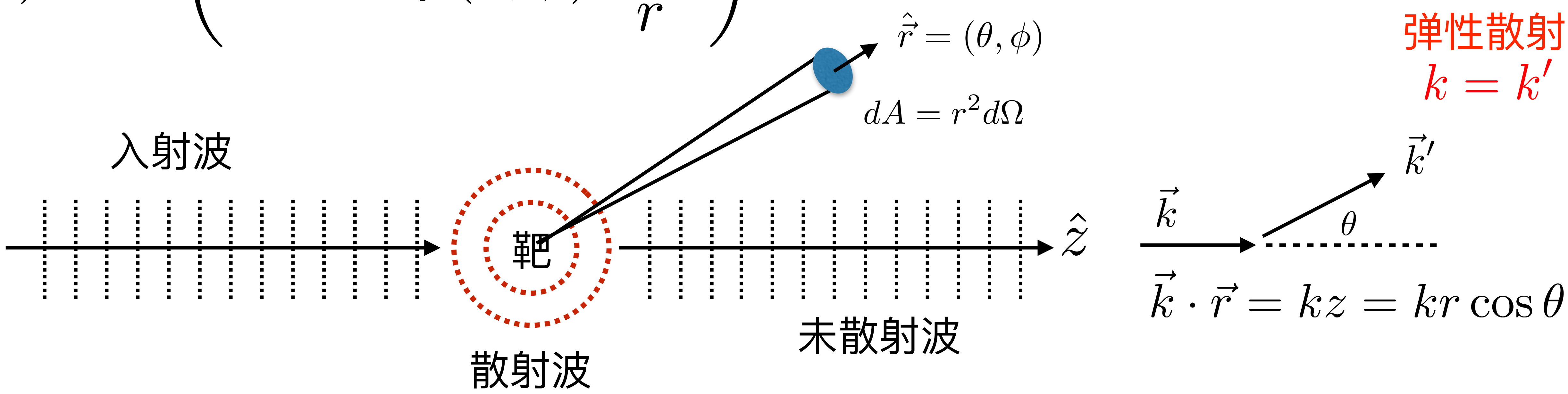
球对称势 $V(\vec{r}) = V(r) \longrightarrow [\hat{H}, \hat{\vec{L}}] = 0 \longrightarrow \hat{L}_{x,y,z}$ 守恒量

入射平面波按照球面波展开

$$e^{i\vec{k}\cdot\vec{r}} = e^{ik_z z} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1) j_{\ell}(kr) P_{\ell}(\cos\theta)$$

问：每个分波（不同 ℓ 值）被 $V(r)$ 改变的情况?

$$\psi(\vec{r}) = A\left(e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}\right)$$



$$\psi(r,\theta) = A\left[\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)j_\ell(kr)P_\ell(\cos\theta) + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$$

探测器远离靶 ($r \to \infty$)： $j_\ell(kr) \xrightarrow{r\to\infty} \dfrac{\sin(kr - \frac{\ell\pi}{2})}{kr}$

$$\psi(r,\theta) \to A\left[\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)P_\ell(\cos\theta)\frac{\sin(kr - \frac{\ell\pi}{2})}{kr} + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$$

$$\sin\left(kr - \frac{\ell\pi}{2}\right) = \frac{1}{2i}\left[e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}\right]$$

$$= \frac{1}{2i}\left[e^{ikr}e^{-i\ell\pi/2} - e^{-ikr}e^{i\ell\pi/2}\right]$$

$$e^{\pm i\frac{\ell\pi}{2}} = \left(e^{\pm i\frac{\pi}{2}}\right)^\ell = (\pm i)^\ell$$

$$= \frac{1}{2i}\left[(-i)^\ell e^{ikr} - i^\ell e^{-ikr}\right]$$

故而

$$\psi(r,\theta) \to -\frac{e^{-ikr}}{2ikr}\sum_{\ell=0}^{\infty} i^{2\ell}(2\ell+1)P_\ell(\cos\theta)$$

$$+\frac{e^{ikr}}{r}\left[f(\theta) + \frac{1}{2ik}\sum_{\ell=0}^{\infty} i^\ell(-i)^\ell(2\ell+1)P_\ell(\cos\theta)\right]$$

求解

定态薛定谔方程的一般解

中心势场中定态SE的一般解形式是

$$\psi(\vec{r}) = \sum_{\ell m} C_{\ell m} R_{k\ell}(r) Y_{\ell m}(\theta, \phi)$$

散射波函数和 ϕ 角无关，可取 $m = 0$

$$\psi(r, \theta) = \sum_{\ell=0} a_\ell R_{k\ell}(r) P_\ell(\cos\theta)$$

其中 $R_{k\ell}(r)$ 满足

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r}\right](rR_{k\ell}) = \frac{2m}{\hbar^2} V(r) \, (rR_{k\ell})$$

分波法求解$f(\theta)$的思路

中心势场中弹性散射过程的波函数：

$$\psi(\vec{r}) = A\left(e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}\right)$$

$$\psi(r,\theta) = \sum_{\ell=0} a_\ell R_{k\ell}(r) P_\ell(\cos\theta)$$

这两个公式描述了同一物理过程，那么它们是完全等价的。

问：是否可以利用$r \to \infty$ 处的渐进行为求解$f(\theta)$

或 $f(\theta) \sim a_\ell$ 之间的关系?

$R_{k\ell}(r)$ 在 $r \to \infty$ 时渐进行为

$r \to \infty$ 时

$$\left(\frac{d^2}{dr^2} + k^2\right)(rR_{k\ell}(r)) = 0$$

通解

$$R_{k\ell}(r) = A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr)$$

$$j_\ell(kr) \xrightarrow{r\to\infty} \frac{\sin(kr - \ell\pi/2)}{kr}$$

$$\eta_\ell(kr) \xrightarrow{r\to\infty} -\frac{\cos(kr - \ell\pi/2)}{kr}$$

$$R_{k\ell}(kr) \xrightarrow{r\to\infty} A_\ell \frac{\sin(kr - \ell\pi/2)}{kr} - B_\ell \frac{\cos(kr - \ell\pi/2)}{kr}$$

$R_{k\ell}(r)$ 在 $r \to \infty$ 时渐进行为

$$R_{k\ell}(kr) \xrightarrow{r\to\infty} A_\ell \frac{\sin(kr - \ell\pi/2)}{kr} - B_\ell \frac{\cos(kr - \ell\pi/2)}{kr}$$

令 $C_\ell^2 = A_\ell^2 + B_\ell^2$，则 $\left.\begin{aligned} A_\ell &= C_\ell \cos\delta_\ell \\ B_\ell &= -C_\ell \sin\delta_\ell \end{aligned}\right\} \quad \tan\delta_\ell = -\frac{B_\ell}{A_\ell}$

其中 $\delta_\ell = -\arctan\left(\frac{B_\ell}{A_\ell}\right)$ 由势场的性质和波函数边界条件确定

$$\begin{aligned} R_{k\ell}(kr) \xrightarrow{r\to\infty} \quad & C_\ell \cos\delta_\ell \frac{\sin(kr - \ell\pi/2)}{kr} \\ & + C_\ell \sin\delta_\ell \frac{\cos(kr - \ell\pi/2)}{kr} \\ & = C_\ell \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)}{kr} \end{aligned}$$

$R_{k\ell}(r)$ 在 $r\to\infty$ 时渐进行为

$$R_{k\ell}(kr) \xrightarrow{r\to\infty} C_\ell \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)}{kr}$$

当V(r)=0 时，$\delta_\ell = 0$，则

$$R_{k\ell}(r) \to C_\ell \frac{\sin(kr - \ell\pi/2)}{kr} \sim j_\ell(kr)$$

此时通解就是平面波按照球面波展开的形式

δ_ℓ： **相移** (phase shift)

它刻画在大 r 处 $R_{k\ell}(r)$ 和 $j_\ell(kr)$的偏离程度

相移是由于V(r)造成的，所以我们猜测 $f(\theta)$ 应该依赖于 δ_ℓ

将 $R_{k\ell}(r)$ 代入到中心势场散射问题的通解

$$\psi(r,\theta)=\sum_{\ell=0}a_\ell R_{k\ell}(r)P_\ell(\cos\theta)$$

我们得到通解波函数在大 r 处的渐进行为

$$\psi(r,\theta)\xrightarrow{r\to\infty}\sum_{\ell=0}a_\ell P_\ell(\cos\theta)\frac{\sin(kr-\frac{\ell\pi}{2}+\delta_\ell)}{kr}$$

形变平面波（和平面波展开式仅仅相差一个相移因子）

因 $$\sin(kr-\frac{\ell\pi}{2}+\delta_\ell)=\frac{1}{2i}\left[(-i)^\ell e^{ikr}e^{i\delta_\ell}-i^\ell e^{-ikr}e^{-i\delta_\ell}\right]$$

故 $$\psi(r,\theta)\to-\frac{e^{-ikr}}{2ikr}\sum_{\ell=0}a_\ell i^\ell e^{-i\delta_\ell}P_\ell(\cos\theta)$$

$$+\frac{e^{ikr}}{2ikr}\sum_{\ell=0}a_\ell(-i)^\ell e^{i\delta_\ell}P_\ell(\cos\theta)$$

通解波函数和总波函数在 $r\to\infty$ 处的渐进行为

通解 $$\psi(r,\theta)\to -\frac{e^{-ikr}}{2ikr}\sum_{\ell=0} a_\ell i^\ell e^{-i\delta_\ell}P_\ell(\cos\theta) + \frac{e^{ikr}}{2ikr}\sum_{\ell=0} a_\ell(-i)^\ell e^{i\delta_\ell}P_\ell(\cos\theta)$$

分波解

$$\psi(r,\theta)\to -\frac{e^{-ikr}}{2ikr}\sum_{\ell=0}^{\infty} i^{2\ell}(2\ell+1)P_\ell(\cos\theta) + \frac{e^{ikr}}{r}\left[f(\theta)+\frac{1}{2ik}\sum_{\ell=0}^{\infty} i^\ell(-i)^\ell(2\ell+1)P_\ell(\cos\theta)\right]$$

1）e^{-ikr} 项系数相等

$$(2\ell+1)i^{2\ell}=a_\ell i^\ell e^{-i\delta_\ell} \longrightarrow a_\ell=(2\ell+1)i^\ell e^{i\delta_\ell}$$

2） e^{ikr} 项系数相等

$$f(\theta)+\frac{1}{2ik}\sum_{\ell=0}^{\infty}i^{\ell}(-i)^{\ell}(2\ell+1)P_{\ell}(\cos\theta)$$

$$=\frac{1}{2ki}\sum_{\ell=0}a_{\ell}(-i)^{\ell}e^{i\delta_{\ell}}P_{\ell}(\cos\theta)$$ $$a_{\ell}=(2\ell+1)i^{\ell}e^{i\delta_{\ell}}$$

$$=\frac{1}{2ki}\sum_{\ell=0}(2\ell+1)i^{\ell}(-i)^{\ell}e^{2i\delta_{\ell}}P_{\ell}(\cos\theta)$$

$$\Rightarrow f(\theta)=\frac{1}{2ki}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\delta_{\ell}}-1\right)P_{\ell}(\cos\theta)$$

$$=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_{\ell}}\sin\delta_{\ell}P_{\ell}(\cos\theta)\equiv\sum_{\ell=0}f_{\ell}(\theta)$$

分波振幅

$$e^{2i\delta_{\ell}}-1=e^{i\delta_{\ell}}(e^{i\delta_{\ell}}-e^{-i\delta_{\ell}})=e^{i\delta_{\ell}}\;2i\sin\delta_{\ell}$$

微分散射截面

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$= \frac{1}{k^2} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) e^{i(\delta_\ell - \delta_{\ell'})}$$

$$\times \sin\delta_\ell \sin\delta_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

总截面

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^\pi |f(\theta)|^2 \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{2\pi}{k^2} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'}$$

$$\times \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta \longrightarrow \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$= \frac{4\pi}{k^2} \sum_{\ell=0} (2\ell+1) \sin^2\delta_\ell$$

$$= \sum_{\ell=0}^{\infty} \sigma_\ell$$

σ_ℓ ： 分波散射截面

光学定理

总散射截面和向前散射振幅 $f(\theta=0)$ 之间关系

因为 $P_\ell(1) = P_\ell(\cos 0) = 1$

故 $$f(\theta=0) = \frac{1}{k}\sum_{\ell}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell$$

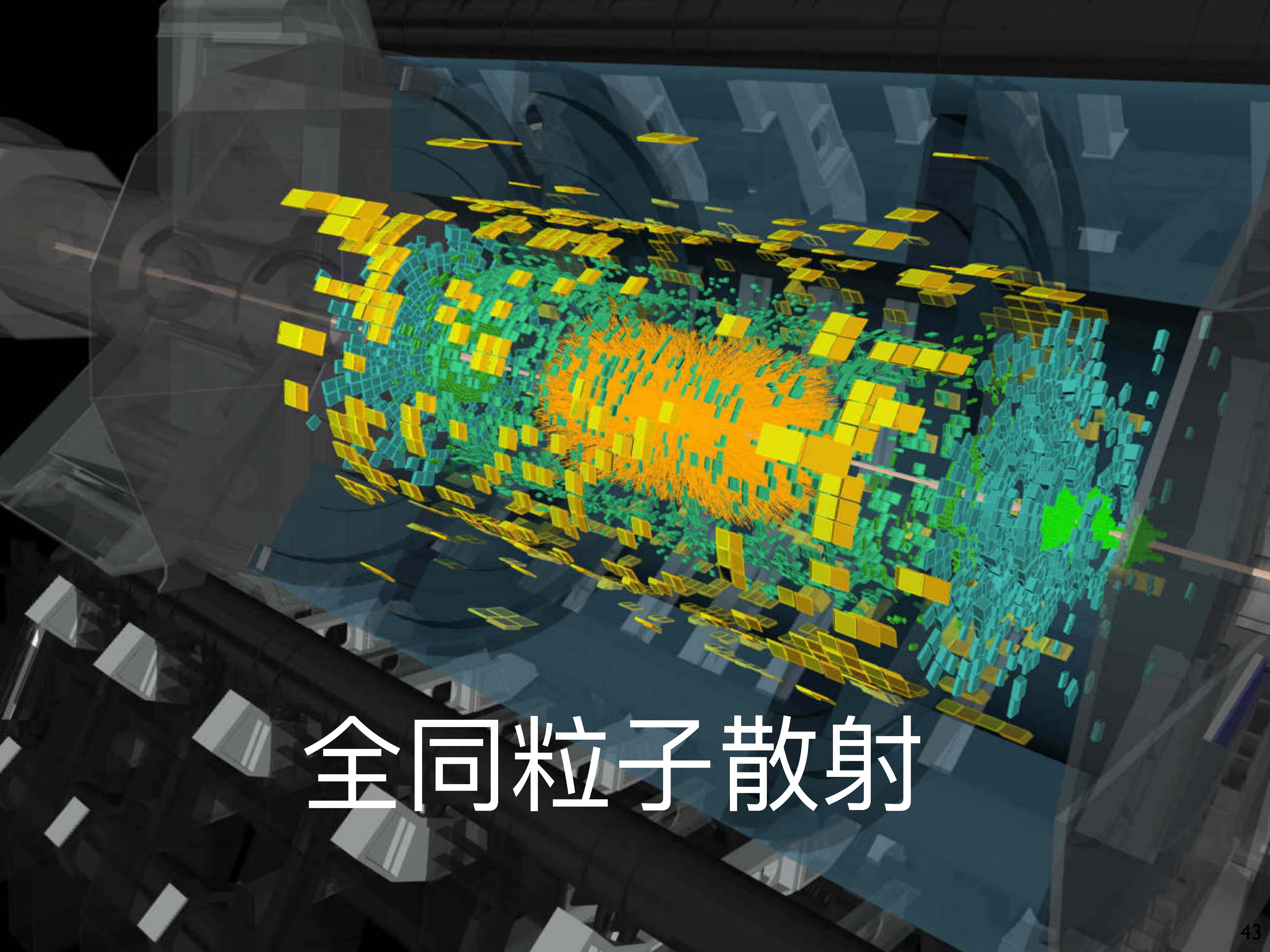
$$= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)(\sin\delta_\ell\cos\delta_\ell + i\sin^2\delta_\ell)$$

又因 $$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

所以 $$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\,\mathrm{Im} f(\theta=0)$$

几率守恒

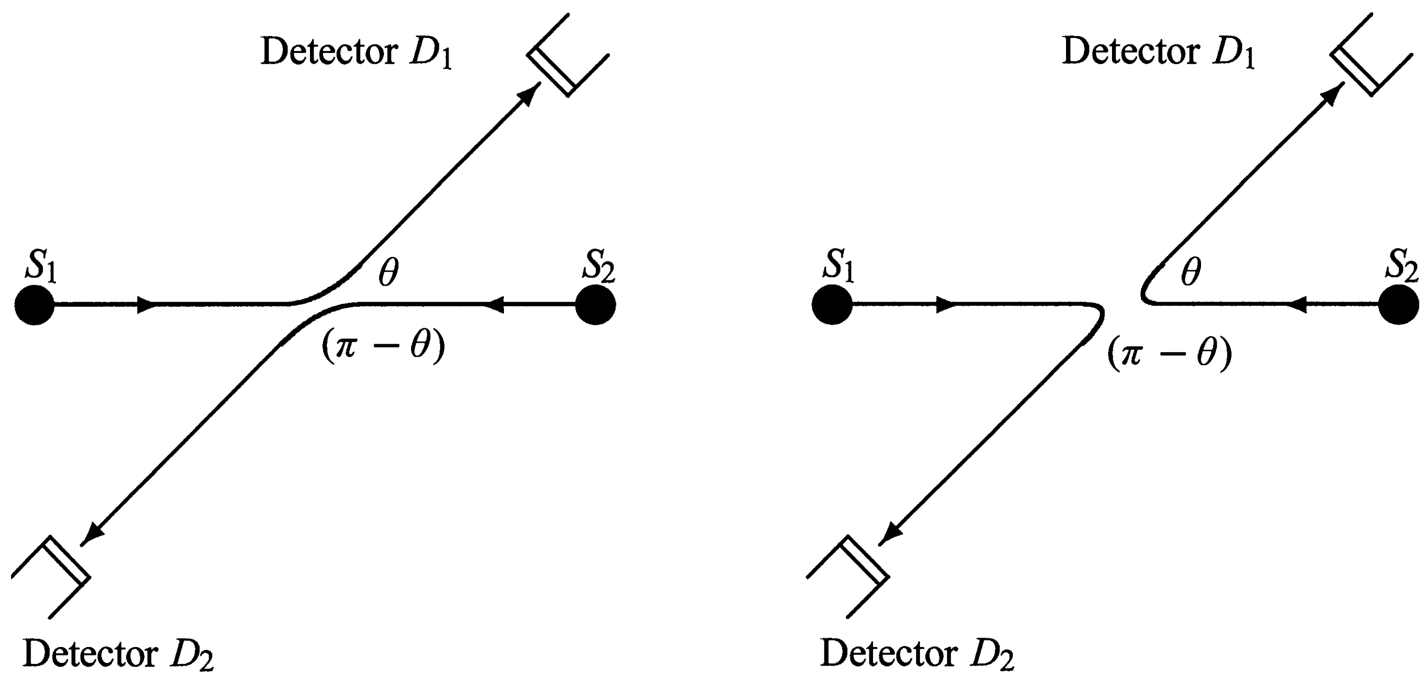
(前向丢失粒子都被散射了)



全同粒子散射

两个全同玻色子散射

经典物理： $\sigma_{\rm cl}(\theta) = \sigma(\theta) + \sigma(\pi - \theta)$



量子物理：无法区分上面两种情况

$$\psi_{\rm sym}(\vec{r}) \to e^{i\vec{k}_0\cdot\vec{r}} + e^{-i\vec{k}_0\cdot\vec{r}} + f_{\rm sym}(\theta)\frac{e^{ikr}}{r}$$

$$\Rightarrow f_{\rm B}(\theta) = f(\theta) + f(\pi - \theta)$$ 对称的散射振幅

两全同玻色子散射微分散射截面

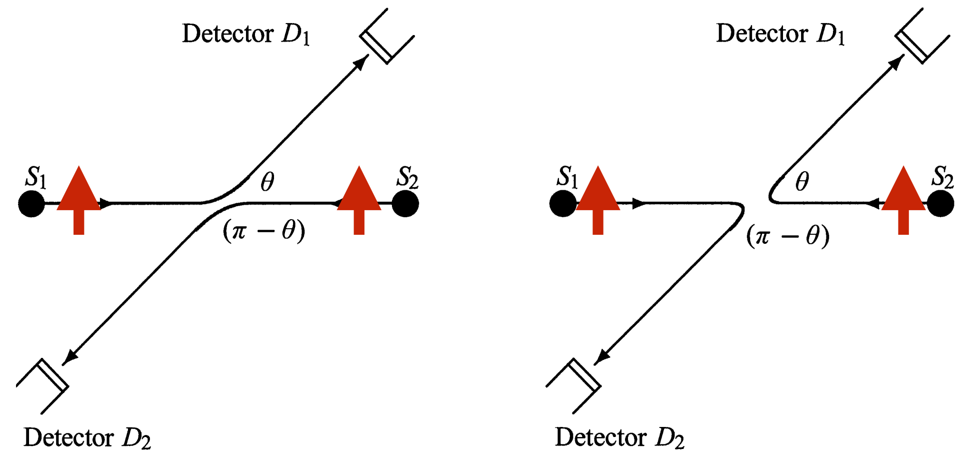
$$f_{\mathrm{B}}(\theta)=f(\theta)+f(\pi-\theta)$$

$$\begin{aligned}\frac{d\sigma_B}{d\Omega}&=|f(\theta)+f(\pi-\theta)|^2\\&=|f(\theta)|^2+|f(\pi-\theta)|^2+f(\theta)^*f(\pi-\theta)+f(\theta)f^*(\pi-\theta)\\&=|f(\theta)|^2+|f(\pi-\theta)|^2+\underbrace{2\Re\left[f(\theta)^*f(\pi-\theta)\right]}_{\text{量子干涉效应}}\end{aligned}$$

$$\theta=\frac{\pi}{2}\quad\begin{cases}\dfrac{d\sigma_B}{d\Omega}=4\left|f\left(\dfrac{\pi}{2}\right)\right|^2 & \text{量子不可区分玻色子对}\\[2ex]\dfrac{d\sigma_{\mathrm{cl}}}{d\Omega}=2\left|f\left(\dfrac{\pi}{2}\right)\right|^2 & \text{经典不可区分玻色子对}\\[2ex]\dfrac{d\sigma_{\mathrm{dif}}}{d\Omega}=\left|f\left(\dfrac{\pi}{2}\right)\right|^2 & \text{经典可区分玻色子对}\end{cases}$$

两个全同费米子散射

空间波函数的对称性质取决于自旋空间波函数



$\uparrow \; s=\frac{1}{2}$

总自旋 S=1: $$\frac{d\sigma_A}{d\Omega}=|f(\theta)-f(\pi-\theta)|^2$$

$$\begin{pmatrix} \uparrow_1\uparrow_2 \\ \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2+\downarrow_1\uparrow_2) \\ \downarrow_1\downarrow_2 \end{pmatrix}$$

总自旋 S=0: $$\frac{d\sigma_S}{d\Omega}=|f(\theta)+f(\pi-\theta)|^2$$

$$\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2-\downarrow_1\uparrow_2)$$

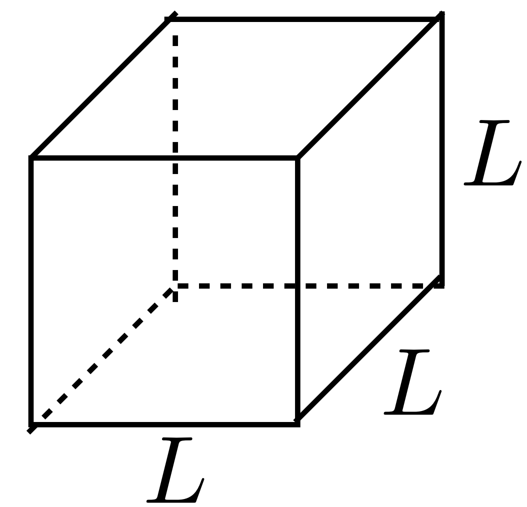
非极化情况: $$\begin{aligned}\frac{d\sigma_{\mathrm{unpol}}}{d\Omega}&=\frac{3}{4}\frac{d\sigma_A}{d\Omega}+\frac{1}{4}\frac{d\sigma_S}{d\Omega}\\&=\frac{3}{4}|f(\theta)-f(\pi-\theta)|^2+\frac{1}{4}|f(\theta)+f(\pi-\theta)|^2\\&=|f(\theta)|^2+|f(\pi-\theta)|^2-\Re[f^*(\theta)f(\pi-\theta)]\end{aligned}$$

附 录

平面波 —— 箱归一化

箱归一化（具有分立谱的动量本征函数）

动量连续谱 $\longrightarrow$ 离散谱 $\xrightarrow[\Delta p \to 0]{}$ 连续谱



设自由粒子处于边长为L的正方形箱子中

不失一般性，考虑x方向。设动量本征函数为 $\psi = Ae^{\frac{ipx}{\hbar}}$

归一化：

$$\int_{L/2}^{L/2} |\psi(x)|^2 \, dx = |A|^2 \int_{L/2}^{L/2} dx = |A|^2 L = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{L}} \Rightarrow \psi_p(x) = \frac{1}{\sqrt{L}} e^{\frac{ipx}{\hbar}}$$

平面波 —— 箱归一化

利用边界条件确定p_x的本征值

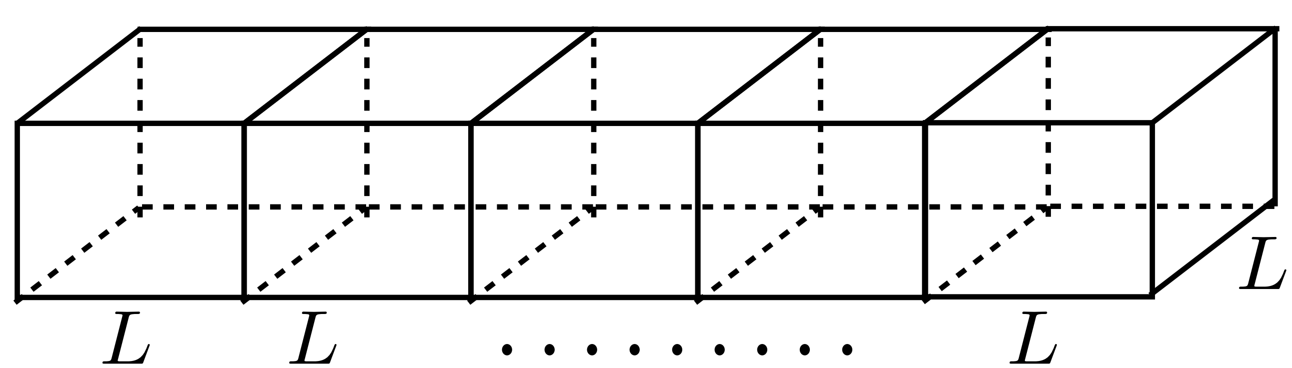
$$\int_{L/2}^{L/2} \psi^* \hat{p}_x \phi dx = -i\hbar(\psi^*\phi)\Big|_{-L/2}^{L/2} + \int_{-L/2}^{L/2} (\hat{p}_x\psi)^* \phi dx$$

$\hat{p}_x$ 的厄米性要求，对任意的 ψ 和 ϕ 都有 $(\psi^*\phi)\Big|_{-L/2}^{L/2} = 0$

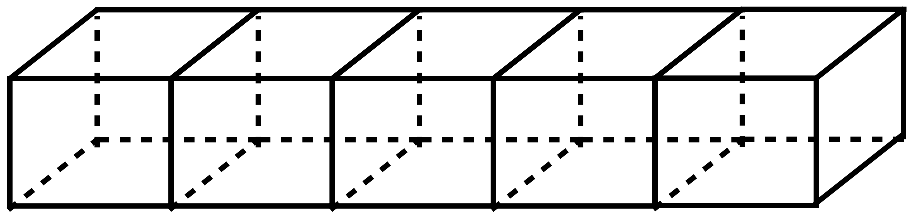
令 $\phi = \psi$ 且设其为 $\hat{p}_x$ 的本征函数，则有

$$|\psi(-L/2)| = |\psi(+L/2)|$$

考虑周期性边界条件 $\psi(-L/2) = \psi(+L/2)$



平面波 —— 箱归一化



$$\psi(-L/2) = \psi(+L/2)$$

周期性边界条件

$$\frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p(-\frac{L}{2})} = \frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p(+\frac{L}{2})} \rightarrow e^{\frac{i}{\hbar} pL} = 1 \rightarrow \frac{pL}{\hbar} = 2n\pi$$

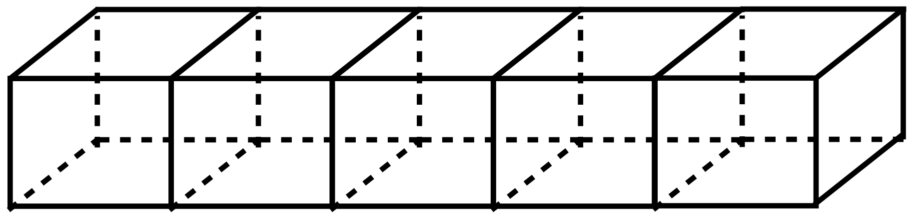
$$\rightarrow \quad p_n = \frac{2\pi\hbar n}{L} = \frac{h}{L} n \qquad n = 0, \pm 1, \pm 2, \cdots$$

故 $\psi_n(x) = \frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p_n x} = \frac{1}{\sqrt{L}} e^{i \frac{2\pi n}{L} x}$

周期性边界条件：p_n是离散的，本征值间隔 $\Delta p_n = \frac{h}{L}$

当 $L \to \infty$ 时，$\Delta p_n \to 0$ 分离谱 —> 连续谱

平面波 —— 箱归一化



$$\psi\left(-L/2\right)=\psi\left(+L/2\right)$$

周期性边界条件

$$\frac{1}{\sqrt{L}}e^{\frac{i}{\hbar}p\left(-\frac{L}{2}\right)}=\frac{1}{\sqrt{L}}e^{\frac{i}{\hbar}p\left(+\frac{L}{2}\right)}\longrightarrow e^{\frac{i}{\hbar}pL}=1\longrightarrow\frac{pL}{\hbar}=2n\pi$$

$$\longrightarrow\quad p_n=\frac{2\pi\hbar n}{L}=\frac{h}{L}n\qquad n=0,\pm1,\pm2,\cdots$$

故 $$\psi_n(x)=\frac{1}{\sqrt{L}}e^{\frac{i}{\hbar}p_nx}=\frac{1}{\sqrt{L}}e^{i\frac{2\pi n}{L}x}$$ 离散的动量本征函数

正交归一性：

$$\int_{-L/2}^{L/2}\psi_n^*(x)\psi_m(x)dx=\frac{1}{L}\int_{-L/2}^{L/2}e^{i\frac{2\pi(n-m)}{L}x}dx=\delta_{nm}$$

箱归一化平面波的封闭性

离散的动量本征函数 $\psi_n(x) = \frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p_n x} = \frac{1}{\sqrt{L}} e^{i \frac{2\pi n}{L} x}$

$$\sum_{n=-\infty}^{+\infty} \psi_m^*(x')\psi_n(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{L} e^{\frac{i}{\hbar} p_n (x-x')}$$

令 $L \to \infty$，$\Delta p_n = \frac{h}{L} \to dp$，即 $\frac{1}{L} \to \frac{dp}{h}$

$$\sum_{n=-\infty}^{+\infty} \cdots \Delta p_n \xrightarrow{L\to\infty} \int_{-\infty}^{+\infty} \cdots dp$$

故 $$\sum_{n=-\infty}^{+\infty} \psi_n^*(x')\psi_n(x) = \sum_{n=-\infty}^{+\infty} e^{\frac{i}{\hbar} p_n (x-x')} \frac{\Delta p_n}{h}$$

$$\xrightarrow{L\to\infty} \int_{-\infty}^{+\infty} \frac{1}{h} e^{\frac{i}{\hbar} p (x-x')} dp = \delta(x-x')$$

满足封闭性

箱归一化平面波的完备性

离散的动量本征函数 $\psi_n(x) = \frac{1}{\sqrt{L}} e^{\frac{i}{\hbar} p_n x} = \frac{1}{\sqrt{L}} e^{i \frac{2\pi n}{L} x}$

$$f(x) = \int_{-\infty}^{+\infty} f(x') \delta(x' - x) dx'$$

$L \to \infty$

$$= \int_{-\infty}^{+\infty} f(x') \left[\sum_{n=-\infty}^{+\infty} \psi_n^*(x') \psi_n(x) \right] dx'$$

$$= \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \psi_n^*(x') f(x') dx' \right] \psi_n(x)$$

$$= \sum_{n=-\infty}^{+\infty} c_n \psi_n(x)$$

满足
完备性

返回