Chapter 7
Harmonic Oscillator Solution Using Operator Methods

We have already discussed the quantum mechanical harmonic oscillator several times in this book including Sections 3.1.2 and 6.5. In this chapter we will examine it yet again, this time using operator formalism, a method that is sometimes characterized as algebraic. We will show that the energy eigenvalues are obtainable without actually solving a differential equation, using only the Hamiltonian operator and the commutation relations between $\hat{x}$ and $\hat{p}$. This powerful method of solution, due to Dirac, has consequences far beyond an exercise in elementary quantum physics. The operators are used in many problems in physics.

7.1 The Algebraic Method

7.1.1 The Schrödinger Picture

We define a new operator in terms of the position and momentum operators $\hat{x}$ and $\hat{p}$. In this section we will be using the Schrödinger picture so the operators are time-independent. Because we are using operator methods, we retain the hat on the position operator. For simplicity we again drop the subscript on the momentum operator inasmuch as this is a one-dimensional problem. The new operator and its Hermitian conjugate are

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} + i\frac{1}{\alpha\hbar}\hat{p}\right) \tag{7.1}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} - i\frac{1}{\alpha\hbar}\hat{p}\right) \tag{7.2}$$

where $\alpha = \sqrt{m\omega/\hbar}$ as in Equation 3.25. Notice that these operators are *not* Hermitian operators. They therefore need not have real eigenvalues and do not qualify as observables. In the form written in Equations 7.1 and 7.2 $\hat{a}$ and $\hat{a}^\dagger$ are dimensionless. To exploit these new operators to solve the TISE we must do some preliminary work to derive relations between them and other quantum mechanical operators.

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We first show that $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$ by invoking the $[\hat{x}, \hat{p}] = i\hbar$:

$$\begin{aligned}
\left[\hat{a}, \hat{a}^{\dagger}\right] &= \frac{1}{2}\left[\left(\alpha\hat{x} + i\frac{1}{\alpha\hbar}\hat{p}\right), \left(\alpha\hat{x} - i\frac{1}{\alpha\hbar}\hat{p}\right)\right] \\
&= \frac{1}{2}\left\{\left(-\frac{i}{\hbar}\right)[\hat{x}, \hat{p}] + \left(\frac{i}{\hbar}\right)[\hat{p}, \hat{x}]\right\} \\
&= \frac{-i}{2\hbar}\{2[\hat{x}, \hat{p}]\} \\
&= 1
\end{aligned} \tag{7.3}$$

Next we express the Hamiltonian in terms of a and $\hat{a}^{\dagger}$ by solving Equations 7.1 and 7.2 for $\hat{x}$ and $\hat{p}$. We obtain

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{\dagger}\right) = \frac{1}{\sqrt{2}\alpha}\left(\hat{a} + \hat{a}^{\dagger}\right) \tag{7.4}$$

and

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^{\dagger}\right) = -i\frac{\alpha\hbar}{\sqrt{2}}\left(\hat{a} - \hat{a}^{\dagger}\right) \tag{7.5}$$

The Hamiltonian then has the simple form

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \\
&= -\frac{\hbar\omega}{4}\left(\hat{a} - \hat{a}^{\dagger}\right)^2 + \frac{\hbar\omega}{4}\left(\hat{a} + \hat{a}^{\dagger}\right)^2 \\
&= \frac{\hbar\omega}{4}\left\{-\left(\hat{a}\hat{a} - \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{a}^{\dagger}\right) + \left(\hat{a}\hat{a} + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{a}^{\dagger}\right)\right\} \\
&= \frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}\right)
\end{aligned} \tag{7.6}$$

We can put this in a slightly different form by adding and subtracting $\hat{a}^{\dagger}\hat{a}$ in the parentheses in this last equation and taking advantage of the commutation relation derived above and $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$. We obtain

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{7.7}$$

In this form the Hamiltonian is the sum of two terms, one the operator $\hat{a}^{\dagger}\hat{a}$ and the other the constant $\hbar\omega/2$. The solution of the eigenvalue problem then becomes one of solving the eigenvalue equation for the operator $\hat{a}^{\dagger}\hat{a}$ and adding the constant term to $\hbar\omega$ times their eigenvalues to find the energies. Note that the operator $\hat{a}^\dagger\hat{a}$ is Hermitian as it must be to be a term in the Hamiltonian. It therefore qualifies as an observable. Moreover, because it obviously commutes with the Hamiltonian, a measurement of this observable can be made simultaneously with a measurement of the energy, and these two operators can have simultaneous eigenvectors.

It is often convenient to express $\hat{a}$ and $\hat{a}^\dagger$ in terms of dimensionless operators. To do this we simply rescale $\hat{x}$ and $\hat{p}$. Using Equations 7.1 and 7.2 as a guide we let

$$\hat{x} = \frac{1}{\alpha}\hat{X} \quad \text{and} \quad \hat{p} = \alpha\hbar\hat{P} \tag{7.8}$$

so that

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right) \tag{7.9}$$

and

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right) \tag{7.10}$$

This choice of scaling constants also preserves the "equality" of $\Delta\hat{X}$ and $\Delta\hat{P}$ as discussed in Section 4.5. Notice that $\hat{X}$ is equivalent to the reduced coordinate $\xi = \alpha x$ of Section 3.1.2.

Now, what is the observable $\hat{a}^\dagger\hat{a}$? Being blessed with knowledge of the answer we will designate this operator with the symbol $\hat{N}$ so that

$$\hat{N} = \hat{a}^\dagger\hat{a} \tag{7.11}$$

and we seek to solve the eigenvalue equation

$$\hat{N}\,|n\rangle = n\,|n\rangle \tag{7.12}$$

where n and $|n\rangle$ are the eigenvalues and eigenvectors of $\hat{N}$ (as well as $\hat{H}$ because $\hat{N}$ and $\hat{H}$ commute). We may write the energy eigenvalue equation in terms of the eigenvectors and eigenvalues of $\hat{N}$:

$$\begin{aligned}\hat{H}\,|n\rangle &= \hbar\omega\left(\hat{N} + \frac{1}{2}\right)|n\rangle \\ &= \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\end{aligned} \tag{7.13}$$

bearing in mind that we are pretending that we do not yet know the nature of the eigenvalue n. One thing we know, however, is that n is unitless because $\hat{a}$ and $\hat{a}^\dagger$ are dimensionless.

We now undertake the crucial task of identifying the nature of the observable $\hat{N}$ by examining the properties of the eigenvalue n. We will require the commutation relations

$$\left[\hat{N}, \hat{a}\right] = -\hat{a} \tag{7.14}$$

and

$$\left[\hat{N}, \hat{a}^{\dagger}\right] = \hat{a}^{\dagger} \tag{7.15}$$

which are easy to prove (see Problem 2). To determine the nature of n, we begin by operating on the vector $\hat{a}\,|n\rangle$ with $\hat{N}$, use Equation 7.14, regroup the operators, and use the commutator in Equation 7.3. We obtain

$$\begin{aligned} \hat{N}\hat{a}\,|n\rangle &= \left(-\hat{a} + \hat{a}\hat{N}\right)|n\rangle \\ \hat{N}\left\{\hat{a}\,|n\rangle\right\} &= \left(-\hat{a} + \hat{a}n\right)|n\rangle \\ &= (n-1)\left\{\hat{a}\,|n\rangle\right\} \end{aligned} \tag{7.16}$$

Equation 7.16 shows two important characteristics of the eigenvalues and eigenfunctions of $\hat{N}$. First, the quantity $\{\hat{a}\,|n\rangle\}$ is also an eigenvector of $\hat{N}$ and, second, its eigenfunction is $(n-1)$. We have therefore deduced a relationship between $\hat{a}\,|n\rangle$ and $|n-1\rangle$, in particular

$$\hat{a}\,|n\rangle = c_1\,|n-1\rangle \tag{7.17}$$

where c_1 is a constant. In a similar manner we operate on $\hat{a}^{\dagger}\,|n\rangle$ with $\hat{N}$ and employ the commutation relation in Equation 7.15 to obtain (see Problem 3)

$$\hat{a}^{\dagger}\,|n\rangle = c_2\,|n+1\rangle \tag{7.18}$$

We must now evaluate the constants c_1 and c_2. To do this we use the fact that these eigenvectors must be normalized. Thus,

$$\begin{aligned} 1 &= \langle n-1\,|n-1\rangle \\ &= \frac{1}{|c_1|^2}\,\langle n|\,\hat{a}^{\dagger}\hat{a}\,|n\rangle \\ &= \frac{1}{|c_1|^2}\,\langle n|\,\hat{N}\,|n\rangle \\ &= \frac{n}{|c_1|^2} \end{aligned} \tag{7.19}$$

It is conventional to choose c_1 to be real in which case $c_1 = \sqrt{n}$ and the effect of $\hat{a}$ operating on $|n\rangle$ is given by

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{7.20}$$

In an analogous manner (see Problem 4) we find that

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{7.21}$$

Equations 7.17 and 7.18 show that the effect of $\hat{a}$ or $\hat{a}^\dagger$ on one of the simultaneous eigenvectors of $\hat{H}$ and $\hat{N}$ is to lower it or raise it (respectively) to the next eigenvector. Equations 7.20 and 7.21 specify the "length" of the vectors that result from the action of each of these operators. The operators $\hat{a}$ and $\hat{a}^\dagger$ are known as ladder operators, raising and lowering operators, or creation and annihilation operators.

Now, what about the nature of n itself? Because the absolute square of $\hat{a}\,|n\rangle$ must be positive we know that

$$\begin{aligned}\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle &= \langle n|\,\hat{N}\,|n\rangle \\ &= n \geq 0\end{aligned} \tag{7.22}$$

Moreover, n cannot be negative so it must be impossible to lower an eigenvector to make its eigenvalue negative. The only way this is possible is if the n are positive integers. Now, clearly, there must be a minimum value of n, call it n_{min}, to avoid negative values. Lowering the eigenket that has this minimum value, call it $|n_{\text{min}}\rangle$, must obliterate it. That is,

$$\hat{a}\,|n_{\text{min}}\rangle = 0 \tag{7.23}$$

Operating on Equation 7.23 with $\hat{a}^\dagger$ leads to

$$\begin{aligned}\hat{a}^\dagger\hat{a}\,|n_{\text{min}}\rangle &= \hat{N}\,|n_{\text{min}}\rangle \\ &= 0\end{aligned} \tag{7.24}$$

which shows that the minimum eigenvalue of $\hat{N}$ must be zero, $n_{\text{min}} = 0$. Thus, n can be any positive integer or zero. For this reason $\hat{N}$ is called the number operator. These restrictions on the eigenvalue n together with Equation 7.13 make it clear that we have recovered the same equation for the energy eigenvalues as we derived by solving the TISE in Section 3.1.2. That is, from Equation 7.13, the energy eigenvalues are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots \tag{7.25}$$

We can also obtain the eigenfunctions given in Equation 3.45 using operator techniques. If we can find the eigenvector corresponding to the lowest eigenvalue, we can simply operate on it with the raising operator $\hat{a}^\dagger$ until we have reached the desired eigenvector. Obviously this is an inferior method to simply looking up the wave function, but it shows that any wave function can be obtained once one of them is known. To obtain the lowest eigenfunction we begin by lowering it out of existence, that is, employing Equation 7.23. To convert eigenkets into eigenfunctions we make use of Equation 6.134 so the ground state eigenfunction $\psi_0(x)$ is $\langle X|0\rangle$. Writing Equation 7.23 in the coordinate representation we have

$$\begin{aligned}\hat{a}\psi_0(x) &= 0\\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{i}{m\omega}\hat{p}\right)\psi_0(x)\\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x)\end{aligned} \tag{7.26}$$

which is a linear first-order differential equation for $\psi_0(x)$. This equation is separable and yields the normalized solution

$$\psi_0(x) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\alpha^2 x^2/2} \tag{7.27}$$

where $\alpha = \sqrt{m\omega/\hbar}$. It must be remembered that an arbitrary harmonic oscillator state vector (wave function) is time-dependent. The time dependence is obtained by operating on $|\psi\rangle$ with the time evolution operator. This has the same effect as multiplying each eigenvector in the expansion of the state vector by the appropriate exponential containing the eigenvalue. In terms of the eigenkets $|n\rangle$, an arbitrary state ket is

$$|\Psi\rangle = \sum_{i=0}^{\infty} a_i e^{-iE_i/\hbar}|i\rangle \tag{7.28}$$

A summary of the relations pertaining to the algebraic solution of the harmonic oscillator is given in Table 7.1.

7.1.2 Matrix Elements

It is useful to know the matrix elements of the ladder operators because powers of $\hat{x}$ and $\hat{p}$ may be written in terms of them (see Table 7.1). From Equations 7.20 and 7.21 it is clear that

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1} \tag{7.29}$$

and

$$\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{m,n+1} \tag{7.30}$$

Table 7.1 Relations involving the raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$, of the harmonic oscillator

$\hat{x} = \frac{1}{\alpha}\hat{X}$
$\hat{p} = \alpha\hbar\hat{P}$
$\hat{a} = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} + i\frac{1}{\alpha\hbar}\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right)$
$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} - i\frac{1}{\alpha\hbar}\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right)$
$\hat{x} = \frac{1}{\sqrt{2}\alpha}\left(\hat{a} + \hat{a}^\dagger\right)$
$\hat{p} = -i\frac{\alpha\hbar}{\sqrt{2}\alpha}\left(\hat{a} - \hat{a}^\dagger\right)$
$\hat{X} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$
$\hat{P} = \frac{-i}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right)$
$\hat{N} = \hat{a}^\dagger\hat{a}$
$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$
$\left[\hat{a}, \hat{a}^\dagger\right] = 1$
$\left[\hat{N}, \hat{a}\right] = -\hat{a} \implies \left[\hat{H}, \hat{a}\right] = -\hbar\omega\hat{a}$
$\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger \implies \left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger$
$\hat{a}\lvert n\rangle = \sqrt{n}\lvert n-1\rangle$
$\hat{a}^\dagger\lvert n\rangle = \sqrt{n+1}\lvert n+1\rangle$

Because we already know $\hat{x}$ and $\hat{p}$ in terms of the ladder operators, Equations 7.4 and 7.5, we can easily find the matrix elements $\langle m|\,\hat{x}\,|n\rangle$ and $\langle m|\,\hat{p}\,|n\rangle$:

$$\begin{aligned}\langle m|\,\hat{x}\,|n\rangle &= \frac{1}{\sqrt{2}\alpha}\langle m|\left(\hat{a} + \hat{a}^\dagger\right)|n\rangle \\ &= \frac{1}{\sqrt{2}\alpha}\left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right)\end{aligned} \tag{7.31}$$

and

$$\begin{aligned}\langle m|\,\hat{p}\,|n\rangle &= -i\frac{\alpha\hbar}{\sqrt{2}}\langle m|\left(\hat{a} - \hat{a}^\dagger\right)|n\rangle \\ &= -i\frac{\alpha\hbar}{\sqrt{2}}\left(\sqrt{n}\delta_{m,n-1} - \sqrt{n+1}\delta_{m,n+1}\right)\end{aligned} \tag{7.32}$$

We can calculate the matrix elements of higher powers of $\hat{x}$ by repetitive application of the ladder operators (see Problem 7). There is, however, another method which, while offering no particular advantage for low powers of $\hat{x}$, is more convenient for higher powers. We illustrate this method by evaluating a low power, $\langle m|\,\hat{x}^2\,|n\rangle$. We write

$$\langle m|\,\hat{x}^2\,|n\rangle = \langle m|\,\hat{x}\hat{x}\,|n\rangle = \sum_{k=0}^{\infty} \langle m|\,\hat{x}\,|k\rangle \, \langle k|\,\hat{x}\,|n\rangle \tag{7.33}$$

where the last step was effected using the identity operator (see Equation 6.58). Inserting the matrix elements from Equation 7.31 we have

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{1}{2\alpha^2}\sum_{k=0}^{\infty}\left[\left(\sqrt{k}\delta_{m,k-1} + \sqrt{k+1}\delta_{m,k+1}\right) \times \left(\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}\right)\right] \tag{7.34}$$

or

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{1}{2\alpha^2}\sum_{k=0}^{\infty}\Big[\sqrt{kn}\delta_{m,k-1}\delta_{k,n-1} + \sqrt{k\,(n+1)}\delta_{m,k-1}\delta_{k,n+1} + \sqrt{n\,(k+1)}\delta_{m,k+1}\delta_{k,n-1} + \sqrt{(n+1)\,(k+1)}\delta_{m,k+1}\delta_{k,n+1}\Big] \tag{7.35}$$

To simplify this expression we must combine the Kronecker deltas, a process that we illustrate by considering the first term on the right-hand side of Equation 7.35. The only term of this summation that survives is the one for which $m = k - 1$ *and* for which $n = k + 1$ so that $m = n - 2$. Then

$$\sum_{k=0}\sqrt{kn}\delta_{m,k-1}\delta_{k,n-1} = \sqrt{n\,(m+1)}\delta_{m+1,n-1} = \sqrt{n\,(n-1)}\delta_{m,n-2} \tag{7.36}$$

where, for the sake of tidiness, we have made the first index on the Kronecker delta m. The remaining terms are treated the same way and we arrive at

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{1}{2\alpha^2}\Big[\sqrt{n\,(n-1)}\delta_{m,n-2} + (2n+1)\,\delta_{m,n} + \sqrt{(n+1)\,(n+2)}\delta_{m,n+2}\Big] \tag{7.37}$$

Similarly, we can obtain $\left(\hat{x}^3\right)_{mn} = \langle m|\,\hat{x}^3\,|n\rangle$. After some labor we arrive at

$$\langle m|\,\hat{x}^3\,|n\rangle = \frac{1}{2\sqrt{2\alpha^3}}\Big[\sqrt{(n+1)(n+2)(n+3)}\delta_{m,n+3} \\ +3\sqrt{(n+1)^3}\delta_{m,n+1} + 3\sqrt{n^3}\delta_{m,n-1} \\ +\sqrt{n(n-1)(n-2)}\delta_{m,n-3}\Big] \tag{7.38}$$

Comparing the two odd powers of $\hat{x}$ with the even powers, we see that $\delta_{m,n}$ occurs only in $\left(\hat{x}^2\right)_{mn}$. This term is missing from the odd powers. This is understandable in terms of the actual eigenfunctions which, recall, have definite parity. Therefore, if $m = n$, the diagonal matrix elements must vanish for the odd powers of $\hat{x}$ while they are present in the even powers.

7.1.3 The Heisenberg Picture

We have already worked out the details of the solution to the harmonic oscillator in the Heisenberg representation in Section 6.5. In that section we obtained the time dependences of the observables $\hat{x}(t)$ and $\hat{p}(t)$. We now show that these solutions can be obtained using the ladder operators.

Although they are not themselves observables because they are not Hermitian, we may insert $\hat{a}$ and $\hat{a}^\dagger$ into the Heisenberg equation of motion, Equation 6.145, to convert them to Heisenberg operators. Of course, we could also apply the unitary transformation using the evolution operator. Using the equation of motion we have

$$\frac{d\hat{a}(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}(t)\right] \tag{7.39}$$

and

$$\frac{d\hat{a}^\dagger(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}^\dagger(t)\right] \tag{7.40}$$

From Problem 1 we know that

$$\left[\hat{H}, \hat{a}(t)\right] = -\hbar\omega\hat{a}(t) \quad \text{and} \quad \left[\hat{H}, \hat{a}^\dagger(t)\right] = \hbar\omega\hat{a}^\dagger(t) \tag{7.41}$$

because commutation relations between Schrödinger operators and Heisenberg operators are invariant (see Problem 7, Chapter 6). We thus have a differential equation for each operator:

$$\frac{d\hat{a}(t)}{dt} = -i\omega\hat{a}(t) \tag{7.42}$$

and

$$\frac{d\hat{a}^{\dagger}(t)}{dt} = i\omega\hat{a}^{\dagger}(t) \tag{7.43}$$

the solutions to which are

$$\begin{aligned}\hat{a}(t) &= \hat{a}(0)\,e^{-i\omega t}\\ \hat{a}^{\dagger}(t) &= \hat{a}^{\dagger}(0)\,e^{i\omega t}\end{aligned} \tag{7.44}$$

Using Equations 7.4 and 7.5 we have

$$\begin{aligned}\left\{\hat{x}(t) + \frac{i}{m\omega}\hat{p}(t)\right\} &= \left\{\hat{x}(0) + \frac{i}{m\omega}\hat{p}(0)\right\}e^{-i\omega t}\\ \left\{\hat{x}(t) - \frac{i}{m\omega}\hat{p}(t)\right\} &= \left\{\hat{x}(0) - \frac{i}{m\omega}\hat{p}(0)\right\}e^{i\omega t}\end{aligned} \tag{7.45}$$

Adding these two equations yields $\hat{x}(t)$ and $\hat{p}(t)$, respectively:

$$\begin{aligned}\hat{x}(t) &= \hat{x}(0)\left\{\frac{e^{-i\omega t} + e^{i\omega t}}{2}\right\} + \frac{1}{m\omega}\hat{p}(0)\left\{\frac{e^{-i\omega t} - e^{i\omega t}}{2i}\right\}\\ &= \hat{x}(0)\cos\omega t + \frac{1}{m\omega}\hat{p}(0)\sin\omega t\\ &= \hat{x}(0)\cos\omega t + \frac{1}{\alpha^2\hbar}\hat{p}(0)\sin\omega t\end{aligned} \tag{7.46}$$

and

$$\begin{aligned}\hat{p}(t) &= \hat{p}(0)\cos\omega t - m\omega\hat{x}(0)\sin\omega t\\ &= \hat{p}(0)\cos\omega t - \alpha\hbar\hat{x}(0)\sin\omega t\end{aligned} \tag{7.47}$$

Thus, the equations of motion for the position and momentum operators that we have derived are identical with those already obtained, Equations 6.157 and 6.158.

We have seen that the commutation rules for Heisenberg operators are the same as those for Schrödinger operators (Problem 7, Chapter 6). How about commutators involving Heisenberg operators at different times? To investigate this we examine the commutator $[\hat{x}(t), \hat{p}(0)]$ for the harmonic oscillator. Using Equation 7.46 we have

$$\begin{aligned}[\hat{x}(t), \hat{p}(0)] &= \left[\left\{\hat{x}(0)\cos\omega t - \frac{1}{\alpha^2\hbar}\hat{p}(0)\sin\omega t\right\}, \hat{p}(0)\right]\\ &= [\hat{x}(0), \hat{p}(0)]\cos\omega t\\ &= i\hbar\cos\omega t\end{aligned} \tag{7.48}$$

For the harmonic oscillator we also have (see Problem 6).

$$[\hat{p}(t), \hat{p}(0)] = -im\omega\hbar \sin \omega t$$
$$[\hat{x}(t), \hat{x}(0)] = -\frac{i\hbar}{m\omega} \sin \omega t \tag{7.49}$$

7.2 Coherent States of the Harmonic Oscillator

After deducing his now-famous equation, Schrödinger searched for a way to relate quantum mechanical parameters to classical physics. In particular, he was looking for a way to quantum mechanically represent the motion of a classical particle. In 1926 he was led to what we may refer to as the Schrödinger coherent state [1]. He found that certain linear combinations of harmonic oscillator eigenfunctions produced Gaussian wave packets that did not spread in time. Moreover, he also noted that if the uncertainties in position and time were equal, as discussed in Section 4.5, the resulting packet would be as close a representation of a classical particle as could be obtained within the bounds of the uncertainty principle.

We have seen in Sections 4.5 and 6.6.1 the uncertainty product for a Gaussian wave packet has the minimum value $\hbar/2$. The ground state of the harmonic oscillator, being a Gaussian form, has minimum uncertainties

$$\Delta x = \frac{1}{\sqrt{2}\alpha} \quad \text{and} \quad \Delta p = \frac{\hbar\alpha}{\sqrt{2}} \tag{7.50}$$

and these uncertainties are equal. Although the ground state of the harmonic oscillator has these magical properties, we cannot relate it to a moving classical particle because it represents a stationary state. We can, however, translate the ground state Gaussian shape to some other equilibrium position, say x_0, and release the particle (for simplicity with zero momentum). Clearly the wave packet will have the same properties as the ground state wave function, but, because this displaced Gaussian is not an eigenstate, it will move in time. This is the displaced ground state discussed in Section 4.5. It is the best that we can do to quantum mechanically simulate the motion of a classical particle because the uncertainties in position and momentum remain minimized *and* equal. We can, in fact, define this Schrödinger coherent state as one for which the uncertainty product $\Delta x \Delta p$ is minimized and for which the individual uncertainties are equal. Note that it is possible to retain the minimum uncertainty product without the condition that the uncertainties are equal. In such a case we have a "squeezed state," a subject of contemporary research.

We have already constructed the Schrödinger coherent state in Section 4.5.3. It is precisely Case III, the Gaussian wave packet under the influence of a harmonic oscillator potential. The initial wave function is given by Equation 4.95 and the time-dependent probability by Equation 4.107. The probability distribution showed clearly that the packet oscillates with the same frequency as the classical oscillator. Moreover, it does not spread in time.

To construct these coherent states we take a clue from Equation 7.23 which exhibits the property of the annihilation operator, that, when it operates on the ground state of the harmonic oscillator, it obliterates it. This equation may be viewed as an eigenvalue equation, a particularly simple one, but an eigenvalue equation nonetheless. That is,

$$\hat{a}\,|0\rangle = 0 \cdot |0\rangle \tag{7.51}$$

Thus, $|0\rangle$, which has Gaussian shape in both coordinate and momentum spaces, is an eigenvector of the lowering operator with eigenvalue zero. This suggests that other eigenvectors of $\hat{a}$ might have similar properties. Indeed they do, but these eigenstates are not eigenstates of the Hamiltonian and the number operator. In general, letting $|z\rangle$ represent an eigenvector of $\hat{a}$, we have

$$\hat{a}\,|z\rangle = \left(\frac{\alpha}{\sqrt{2}}\zeta\right)|z\rangle \tag{7.52}$$

where the eigenvalue ζ need not be real because $\hat{a}$ is not Hermitian. The factor $\alpha/\sqrt{2}$ has been inserted to make the result dimensionally appealing when compared with Equation 4.107.

To investigate the nature of these eigenvalues we solve Equation 7.52 in coordinate space. In this case $\langle x|\,z\rangle = \Psi(x,0)$ and we have the differential equation

$$\sqrt{\frac{1}{2}}\left[\alpha x + i\frac{1}{\alpha\hbar}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\right]\Psi(x,0) = \frac{\alpha}{\sqrt{2}}\zeta\Psi(x,0) \tag{7.53}$$

the solution to which is

$$\Psi(x,0) = Me^{-\alpha^2(x-\zeta)^2/2} \tag{7.54}$$

where M is the normalization constant. Recalling that the eigenvalue ζ may be a complex number, we let $\zeta = \text{Re}\,\zeta + i\,\text{Im}\,\zeta$ and normalize. Equation 7.54 is

$$\begin{aligned}\Psi(x,0) &= M\exp\left[-\frac{\alpha^2}{2}\left(x - \text{Re}\,\zeta - i\,\text{Im}\,\zeta\right)^2\right]\\ &= M\exp\left\{-\frac{\alpha^2}{2}\left[(x-\text{Re}\,\zeta)^2 - (\text{Im}\,\zeta)^2\right]\right\}\cdot\exp\left[i\alpha^2(x-\text{Re}\,\zeta)\cdot\text{Im}\,\zeta\right]\end{aligned} \tag{7.55}$$

Then

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} \Psi^* (x,0)\, \Psi (x,0)\, dx \\ &= |M|^2 \int_{-\infty}^{\infty} \exp\left\{-\alpha^2 \left[(x - \operatorname{Re}\zeta)^2 - (\operatorname{Im}\zeta)^2\right]\right\} dx \\ &= |M|^2 \exp\left[\alpha^2 (\operatorname{Im}\zeta)^2\right] \frac{\sqrt{\pi}}{\alpha}\end{aligned} \tag{7.56}$$

so that

$$M = \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\frac{\alpha^2}{2} (\operatorname{Im}\zeta)^2\right] \tag{7.57}$$

The eigenfunction represented by Equation 7.54 makes it clear that the eigenvectors of the lowering operator lead to nonstationary states that are nonetheless Gaussian wave packets centered at $x = \operatorname{Re}\zeta$ at $t = 0$. Moreover, the uncertainty product $\Delta x \Delta p$ is minimized and the individual uncertainties are equal. They are not eigenstates of the Hamiltonian and the number operator, but they may be expanded in terms of their eigenstates. These displaced ground states are indeed the Schrödinger coherent states.

To write the eigenket of $\hat{a}$ in terms of the harmonic oscillator eigenkets $|n\rangle$ we write the usual expansion on the complete set

$$|z\rangle = \sum_{n=0}^{\infty} b_n\, |n\rangle \tag{7.58}$$

Applying the annihilation operator to Equation 7.58, using Equation 7.52 and Equation 7.20, we have

$$\begin{aligned}\frac{\alpha}{\sqrt{2}} \zeta\, |z\rangle &= \sum_{n=0}^{\infty} b_n \hat{a}\, |n\rangle \\ &= \sum_{n=1}^{\infty} b_n \sqrt{n}\, |n-1\rangle\end{aligned} \tag{7.59}$$

We now replace $|z\rangle$ with the expansion of Equation 7.58, but change the index on the summation on the right-hand side by letting $n \to (n+1)$ which makes the two summations have compatible ranges, namely, $0 \to \infty$.

$$\frac{\alpha}{\sqrt{2}} \zeta \sum_{n=0}^{\infty} b_n\, |n\rangle = \sum_{n=0}^{\infty} b_{n+1} \sqrt{n+1}\, |n\rangle \tag{7.60}$$

Because the summations are identical, the coefficients of $|n\rangle$ must be identical and we obtain the recursion relation

$$\frac{\alpha}{\sqrt{2}}\zeta b_n = b_{n+1}\sqrt{n+1} \quad \Rightarrow \quad b_{n+1} = \frac{\alpha}{\sqrt{2}}\frac{\zeta}{\sqrt{n+1}}b_n \tag{7.61}$$

Applying this recursion relation n times to the first expansion coefficient b_0 we have

$$\begin{aligned} b_1 &= \left(\frac{\alpha}{\sqrt{2}}\right)\left(\frac{\zeta}{\sqrt{1}}b_0\right) \\ b_2 &= \left(\frac{\alpha}{\sqrt{2}}\right)\frac{\zeta}{\sqrt{2}}b_1 = \left(\frac{\alpha}{\sqrt{2}}\right)^2\left(\frac{\zeta^2}{\sqrt{1\cdot 2}}b_0\right) \\ &\vdots \\ b_n &= \left(\frac{\alpha}{\sqrt{2}}\right)^n\left(\frac{\zeta^n}{\sqrt{n!}}\right)b_0 \end{aligned} \tag{7.62}$$

which we can now insert in Equation 7.58 to obtain

$$|z\rangle = b_0\sum_{n=0}^{\infty}\left(\frac{\alpha}{\sqrt{2}}\right)^n\frac{\zeta^n}{\sqrt{n!}}|n\rangle \tag{7.63}$$

To find b_0 we normalize $|z\rangle$. Taking advantage of the orthonormality of the $|n\rangle$ we have

$$\begin{aligned} \langle z\,|z\rangle &= |b_0|^2\sum_{n=0}^{\infty}\left(\frac{\alpha^2}{2}\right)^n\frac{(\zeta^*)^n\,\zeta^n}{n!} \\ &= |b_0|^2\sum_{n=0}^{\infty}\left(\frac{\alpha^2\,|\zeta|^2}{2}\right)^n\frac{1}{n!} \end{aligned} \tag{7.64}$$

The summation is the Taylor series for $e^{\alpha^2|\zeta|^2/2}$ so b_0 is given by

$$b_0 = e^{-\alpha^2|\zeta|^2/4} \tag{7.65}$$

and

$$b_n = \left(\frac{\alpha}{\sqrt{2}}\right)^n\frac{\zeta^n}{\sqrt{n!}}e^{-\alpha^2|\zeta|^2/4} \tag{7.66}$$

The coherent state, Equation 7.58, is therefore

$$|z\rangle = e^{-\alpha^2|\zeta|^2/4}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\sqrt{2}}\right)^n\zeta^n\frac{1}{\sqrt{n!}}|n\rangle \tag{7.67}$$

which may be shown to be an eigenvector of $\hat{a}$ with eigenvalue $\left(\alpha\zeta/\sqrt{2}\right)$(see Problem 12).

We may now put in the time dependence by applying the time evolution operator to $|z\rangle$ which has the effect of multiplying each term in the expansion of Equation 7.67 by $\exp\left[-i\left(n+1/2\right)\omega t\right]$. Thus, the coherent state including the time dependence is

$$|z\left(t\right)\rangle = e^{-\alpha^2|\zeta|^2/4}e^{-i\omega t/2}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\sqrt{2}}\right)^n\left(\zeta e^{-i\omega t}\right)^n\frac{1}{\sqrt{n!}}|n\rangle \tag{7.68}$$

It is seen that, to obtain the time dependence of the Schrödinger coherent state, it is only necessary to make the substitution $\zeta \rightarrow \zeta e^{-i\omega t}$ in Equation 7.67 and multiply the entire expression by the phase factor representing the zero point energy of the oscillator, $e^{-i\omega t/2}$. Notice that, as remarked in Section 4.5, this is only possible because the energy levels of the harmonic oscillator are equally spaced.

We know that initially the wave packet was a minimum uncertainty packet. That is,

$$\Delta x\Delta p = \frac{\hbar}{2} \tag{7.69}$$

so we can examine the time evolution of the individual uncertainties by finding the uncertainties for $t = 0$ and then make the substitution $\zeta \rightarrow \zeta e^{-i\omega t}$ in $\Psi\left(x,0\right)$. We begin by calculating Δp and leave the determination of Δx as an exercise (see Problem 15).

The expectation values of $\hat{p}$ and $\hat{p}^2$ are conveniently calculated using the ladder operators, Equation 7.52, and $\hat{p}$ in the form given in Equation 7.5. We have

$$\begin{aligned}\langle\hat{p}\rangle &= \left(-i\frac{\alpha\hbar}{\sqrt{2}}\right)\langle z|\left(\hat{a}-\hat{a}^\dagger\right)|z\rangle \\ &= \left(-i\frac{\alpha\hbar}{\sqrt{2}}\right)\left(\langle z|\left(\hat{a}\,|z\rangle - \langle z|\,\hat{a}^\dagger\right)|z\rangle\right) \\ &= \left(-i\frac{\alpha\hbar}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}}\right)\left(\zeta-\zeta^*\right) \\ &= \left(-i\frac{\alpha^2\hbar}{2}\right)2i\,\mathrm{Im}\,\zeta \\ &= \alpha^2\hbar\,\mathrm{Im}\,\zeta\end{aligned} \tag{7.70}$$

where the second inner product was performed using the complex conjugate of Equation 7.52. We also require $\left\langle\hat{p}^2\right\rangle$ which may be calculated in an analogous manner:

$$
\begin{aligned}
\langle \hat{p}^2 \rangle &= \left(-i\frac{\alpha\hbar}{\sqrt{2}}\right)^2 \langle z|\left(\hat{a}-\hat{a}^\dagger\right)\left(\hat{a}-\hat{a}^\dagger\right)|z\rangle \\
&= -\left(\frac{\alpha\hbar}{\sqrt{2}}\right)^2 \langle z|\left[\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \left(\hat{a}^\dagger\right)^2\right]|z\rangle
\end{aligned}
\tag{7.71}
$$

We know that the inner product $\langle z|\,\hat{a}^\dagger\hat{a}\,|z\rangle = \zeta^*\zeta\alpha^2/2$, but how about $\langle z|\,\hat{a}\hat{a}^\dagger\,|z\rangle$? We can easily evaluate this using the commutation relation given in Equation 7.3. Replacing the operator $\hat{a}\hat{a}^\dagger$ with $\left(1+\hat{a}^\dagger\hat{a}\right)$ leads to

$$
\begin{aligned}
\langle \hat{p}^2 \rangle &= -\left(\frac{\alpha\hbar}{\sqrt{2}}\right)^2 \langle z|\left[\hat{a}^2 - 1 - 2\hat{a}^\dagger\hat{a} - \left(\hat{a}^\dagger\right)^2\right]|z\rangle \\
&= -\left(\frac{\alpha\hbar}{\sqrt{2}}\right)^2 \left(\zeta^2 - 2\zeta^*\zeta + \zeta^{*2}\right)\left(\frac{\alpha}{\sqrt{2}}\right)^2 + \left(\frac{\alpha\hbar}{\sqrt{2}}\right)^2 \\
&= -\left(\frac{\alpha^4\hbar^2}{4}\right)(2i\,\mathrm{Im}\,\zeta)^2 + \frac{\alpha^2\hbar^2}{2} \\
&= \alpha^4\hbar^2\,(\mathrm{Im}\,\zeta)^2 + \frac{\alpha^2\hbar^2}{2}
\end{aligned}
\tag{7.72}
$$

Therefore,

$$
\begin{aligned}
(\Delta p)^2 &= \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \\
&= \alpha^4\hbar^2\,(\mathrm{Im}\,\zeta)^2 + \frac{\alpha^2\hbar^2}{2} - \left(\alpha^2\hbar\,\mathrm{Im}\,\zeta\right)^2 \\
&= \frac{\alpha^2\hbar^2}{2}
\end{aligned}
\tag{7.73}
$$

Notice that it is not possible to make the substitution $\zeta \rightarrow \zeta e^{-i\omega t}$ in the expression for $(\Delta p)^2$, Equation 7.73, in order to determine its time dependence since it is independent of ζ and therefore independent of time.

In a similar manner we find (see Problem 16)

$$
\langle \hat{x} \rangle = \mathrm{Re}\,\zeta \quad \text{and} \quad \langle \hat{x}^2 \rangle = (\mathrm{Re}\,\zeta)^2 + \frac{1}{2\alpha^2}
\tag{7.74}
$$

so that

$$
(\Delta x)^2 = \frac{1}{2\alpha^2}
\tag{7.75}
$$

which is also independent of time. The uncertainty product is therefore given *at all times* by that of the minimum uncertainty wave packet

$$(\Delta x)^2 (\Delta p)^2 = \frac{\hbar^2}{4} \tag{7.76}$$

Comparison of the values obtained for the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$, Equations 7.70 and 7.74, leads to a physical interpretation of the complex eigenvalue $\left(\alpha/\sqrt{2}\right)\zeta$. From Equations 4.58 and 4.59 for a Gaussian wave packet with initial displacement x_0 and initial momentum p_0, we know that $\langle \hat{x} \rangle = x_0$ and $\langle \hat{p} \rangle = p_0$ (see Problem 4 of Chapter 4). We are therefore led to write

$$\begin{aligned} \zeta &= \mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta \\ &= \langle \hat{x} \rangle + i \frac{\langle \hat{p} \rangle}{\alpha^2 \hbar} \\ &= x_0 + i \frac{p_0}{\alpha^2 \hbar} \end{aligned} \tag{7.77}$$

so that

$$\mathrm{Re}\,\zeta = x_0 \quad \text{and} \quad \mathrm{Im}\,\zeta = \frac{p_0}{\alpha^2 \hbar} \tag{7.78}$$

Inserting ζ in this form into Equation 7.55 we may write $\Psi(x, 0)$ in the form

$$\Psi(x, 0) = M \exp\left\{-\frac{\alpha^2}{2}\left[(x - x_0)^2 - \left(\frac{p_0}{\alpha^2 \hbar}\right)^2\right]\right\} \cdot \exp\left[i\alpha^2 (x - x_0) \cdot \frac{p_0}{\alpha^2 \hbar}\right] \tag{7.79}$$

where, using Equations 7.57 and 7.78, the normalization constant is

$$M = \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\frac{1}{2}\frac{p_0{}^2}{\alpha^2 \hbar}\right] \tag{7.80}$$

It was remarked in Section 6.6.2 that we would reexamine the time development of a Gaussian packet under the influence of a harmonic oscillator potential, an exercise that will now be undertaken. We wish to find the time-dependent wave function $\Psi(x, t)$ so we may compare $|\Psi(x, t)|^2$ with the corresponding probability density of Case III of Section 4.5 Equation 4.107 to verify that the wave packet of Section 4.5 is indeed a Schrödinger coherent state. To obtain the time dependence of a coherent state it is natural to begin by applying the time evolution operator, Equation 6.115, so that in the current notation

$$\Psi(x, t) = e^{-i\hat{H}t/\hbar}\Psi(x, 0) \tag{7.81}$$

As noted in Section 6.6.2, however, this is a nontrivial exercise because the Hamiltonian consists of two noncommuting terms (see the BCH formula in Appendix L). We can circumvent the necessity of using the time evolution operator because we found that $|z\rangle$ could be converted to $|z(t)\rangle$ by making the substitution $\zeta \to \zeta e^{-i\omega t}$

and multiplying the entire expression by $e^{-i\omega t/2}$ (see Equation 7.68). We therefore take that approach here to effect the conversion $\Psi(x,0) \rightarrow \Psi(x,t)$. Making the substitution $\zeta \rightarrow (\mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta)\,e^{-i\omega t}$ in Equation 7.79 and substituting for the real and imaginary parts of ζ using Equation 7.78 we have

$$\Psi(x,t) = Ne^{-i\omega t/2} \exp\left\{-\frac{\alpha^2}{2}\left[\left(x - x_0 e^{-i\omega t}\right)^2 - \left(\frac{p_0}{\alpha^2\hbar}e^{-i\omega t}\right)^2\right]\right\} \times \exp\left[-2i\left(x - x_0 e^{-i\omega t}\right)\frac{p_0}{\alpha^2\hbar}e^{-i\omega t}\right] \tag{7.82}$$

Adapting Equation 7.82 to the conditions of Case III of Section 4.5, namely, initial momentum $p_0 = 0$, we have

$$\Psi(x,t) = \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left\{\left[-\frac{\alpha^2}{2}(x - x_0\cos\omega t)^2\right]\right\} \times \exp\left[-i\,(x - x_0\cos\omega t)\,x_0\sin\omega t\right] \tag{7.83}$$

and the probability distribution is

$$|\Psi(x,t)|^2 = \frac{\alpha}{\sqrt{\pi}}\exp\left[-\alpha^2(x - x_0\cos\omega t)^2\right] \tag{7.84}$$

We see that, indeed, we have reproduced the probability distribution of Equation 4.107 so that the Schrödinger coherent state of this section is identical with the Gaussian wave packet under the influence of a harmonic oscillator potential.

7.3 Retrospective

The algebraic method of solving the harmonic oscillator problem is a concise and elegant method of obtaining the energy eigenvalues and eigenfunctions. It provides a method of solving quantum mechanical problems using the commutator rules that are a unique feature of quantum physics. Because the harmonic oscillator is the starting point for the quantal description of a great many physical problems, for example, molecular vibrations and nuclear structure, the formulation in terms of the ladder operators is invaluable. Moreover, ladder operators of other operators are employed throughout quantum physics.

7.4 Reference

1. E. Schrödinger, Naturwissenschaften, **28**, 664-666 (1926). The English translation of this paper is contained in E. Schrödinger, Collected Papers on Wave Mechanics (Chelsea Publishing Co., New York, 3rd ed., 1982), pp. 41–44.

Problems

1. Show that $\left[\hat{H}, \hat{a}\right] = -\hbar\omega\hat{a}$ and $\left[\hat{H}, \hat{a}^{\dagger}\right] = \hbar\omega\hat{a}^{\dagger}$.
2. Prove Equations 7.14 and 7.15. That is, prove $\left[\hat{N}, \hat{a}\right] = -\hat{a}$ and $\left[\hat{N}, \hat{a}^{\dagger}\right] = \hat{a}^{\dagger}$.
3. Show that $\hat{a}^{\dagger} |n\rangle = c_2 |n + 1\rangle$.
4. Beginning with $\hat{a}^{\dagger} |n\rangle = c_2 |n + 1\rangle$, show that $\hat{a}^{\dagger} |n\rangle = \sqrt{n+1}\, |n + 1\rangle$.
5. The state vector at $t = 0$ for a particle subject to a harmonic oscillator potential is given by

$$|\Psi(x, 0)\rangle = \frac{1}{\sqrt{3}} |1\rangle + \sqrt{\frac{2}{3}} |2\rangle$$

   where the $|n\rangle$ are eigenvectors of the Hamiltonian and the number operator.

   (a) Find the state vector as a function of time $|\Psi(x, t)\rangle$.
   (b) Find the expectation value of the energy as a function of time.
   (c) Find the expectation value of the position as a function of time.

6. Show that for the harmonic oscillator $[\hat{p}(t), \hat{p}(0)] = -im\omega\hbar \sin\omega t$ and $[\hat{x}(t), \hat{x}(0)] = -\dfrac{i\hbar}{m\omega} \sin\omega t$.
7. Show that

$$\langle m|\, \hat{x}^2 \,|n\rangle = \frac{1}{2\alpha^2} \Big[\sqrt{n(n-1)}\delta_{m,n-2} + (2n+1)\,\delta_{m,n} + \sqrt{(n+1)(n+2)}\delta_{m,n+2}\Big]$$

   by writing x in terms of the ladder operators and operating on $|n\rangle$.
8. Obtain the matrix element $\langle m|\, \hat{x} \,|n\rangle$, Equation 7.31, by direct integration using the wave functions given in Equation 3.49.
9. Obtain the matrix element $\langle m|\, \hat{x}^2 \,|n\rangle$ in terms of the quantum number m by applying the same technique to Equation 7.37 as that employed to obtain Equation 7.35 . Of course, the answer will be the same as that in Problem 7, but show that the δ-functions at each end interchange. [Hint: Make sure that the second index in all the δ-functions is n.]
10. Obtain the matrix element $\langle m|\, \hat{x}^3 \,|n\rangle$ Equation 7.38:

$$\langle m|\, \hat{x}^3 \,|n\rangle = \frac{1}{2\sqrt{2}\alpha^3} \Big[\sqrt{(n+1)(n+2)(n+3)}\delta_{m,n+3} + 3\sqrt{(n+1)^3}\delta_{m,n+1} + 3\sqrt{n^3}\delta_{m,n-1} + \sqrt{n(n-1)(n-2)}\delta_{m,n-3}\Big]$$

11. Find the expectation value of $\hat{x}^4$ for arbitrary state of the harmonic oscillator.
12. Show that the time-dependent Schrödinger coherent state in the form given in Equation 7.68

$$|z(t)\rangle = e^{-\alpha^2|z|^2/4} e^{-i\omega t/2} \sum_{n=0}^{\infty} \left(\frac{\alpha z e^{-i\omega t}}{\sqrt{2}} \right)^n \frac{1}{\sqrt{n!}} |n\rangle$$

is an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue $\left(\alpha z e^{-i\omega t}/\sqrt{2}\right)$ which shows that $|z(t)\rangle$ remains an eigenvector of $\hat{a}$ for all time.
13. Show that the wave function at $t = 0$ for Case III in Chapter 4, Equation 4.95, is an eigenstate of the annihilation operator with eigenvalue $\alpha x_0/\sqrt{2}$.
14. Show that the Schrödinger coherent state

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

can be written in the form

$$|z\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger} |0\rangle$$

15. Show that $(\Delta x)^2 = 1/2\alpha^2$ for the Schrödinger coherent state of Equation 7.67.
16. Show that Δx is independent of time using Equations 7.44.
17. A particle of mass m is in the ground state of a harmonic oscillator potential $U(x) = (1/2)m\omega^2$. At $t = 0$ the force center is suddenly shifted to a point along the x-axis $x = x_0$. The shift is so sudden that the wave function does not change.
    (a) Show that the state of the system after the shift is a Schrödinger coherent state.
    (b) If the energy is measured immediately after the change, what values of the energy can be measured and with what probabilities?