exactly (3.90). The half-space potential has the same eigenfunctions and eigenvalues as the odd-parity solutions to the symmetric potential.

3.3 Electron Tunneling

WKBJ is often used to calculate the tunneling rate of particles through potential barriers. Of course, it is better to use exact eigenfunctions for the evanescent waves, but they are often unavailable. WKBJ is usually an accurate method of obtaining the tunneling rates.

A potential barrier has $V(x) > E$. For this case the momentum is an imaginary variable. The amplitude of the eigenfunction in the barrier has the form

$$\psi(x) = T_0 \exp[-\alpha(x)] \tag{3.91}$$

$$\alpha(x) = \frac{\sqrt{2m}}{\hbar} \int_0^x dx' \sqrt{V(x') - E} \tag{3.92}$$

The prefactor is usually taken to be a constant T_0, with no momentum term. The barrier is considered to be over the interval $0 < x < L$, and the phase integral is started at $x = 0$. There is never a phase constant such as $\pi/4$ for the evanescent wave. The tunneling probability is the absolute magnitude squared of the eigenfunction at $x = L$:

$$P = |\psi(L)|^2 = |T_0|^2 \exp[-2\alpha(L)] \tag{3.93}$$

The first case is a simple repulsive square well of height $V_0 > 0$ and width L. The tunneling probability is

$$\alpha(L) = \gamma L, \quad \gamma = \frac{\sqrt{2m}}{\hbar} \sqrt{V_0 - E} \tag{3.94}$$

$$P = |\psi(L)|^2 = |T_0|^2 \exp[-2\gamma L] \tag{3.95}$$

The same result is found when solving the exact eigenfunction.

The next case is called *Fowler-Nordheim* tunneling. The usual experimental geometry is to put a positive voltage on the surface of a metal to assist electrons to exit the surface. Figure 3.6 shows the surface region. The shaded region on the left shows the occupied electron states $E < \mu$, where μ is the chemical potential. The electrons are confined to the metal by a step potential that has a work function $e\phi$ from the chemical potential. The external potential is represented by an electric field E that makes $F = eE$. The potential function is $V(x) = e\phi - Fx$, where its zero is defined as the chemical potential. The tunneling exponent is

$$\alpha(x) = \frac{\sqrt{2m}}{\hbar} \int_0^x dx' \sqrt{e\phi - Fx' - E} \tag{3.96}$$

The electron exits the triangular barrier at the point $x' = L = (e\phi - E)/F$. The integral is similar to those for the linear potential

$$\alpha(L) = \frac{2}{3} \frac{\sqrt{2m}}{\hbar F} [e\phi - E]^{3/2} \tag{3.97}$$

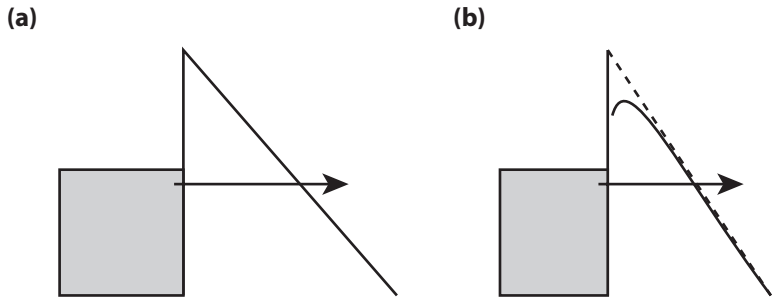

FIGURE 3.6. Fowler-Nordheim tunneling. (a) Electrons must tunnel through a triangular barrier. (b) The addition of the image potential reduces the barrier height.

$$P = |T_0|^2 \exp[-2\alpha(L)] \tag{3.98}$$

The interesting feature is that the tunneling rate has an exponent that is inversely proportional to the applied field F. At large values of the electric field, the exponent becomes small and the tunneling is very rapid. The factor $e\phi - E$ must be positive or there is no need to tunnel.

When an electron is outside of the surface of a perfect conductor, it has an image potential $-e^2/4x$. A better theory of Fowler-Nordheim tunneling includes this image potential:

$$V(x) = e\phi - Fx - \frac{e^2}{4x} \tag{3.99}$$

This potential is shown in figure 3.6b. There are now two turning points $b_{L,R}$ that are both positive. They are found as the points where $V(b) = E$:

$$0 = e\phi - Fb - E - \frac{e^2}{4b} \tag{3.100}$$

$$b_j = \xi \pm \sqrt{\xi^2 - e^2/4F}, \quad \xi = \frac{e\phi - E}{2F} \tag{3.101}$$

$$\alpha = \frac{\sqrt{2mF}}{\hbar^2} \int_{b_L}^{b_R} \frac{dx'}{\sqrt{x'}} \sqrt{(b_R - x')(x' - b_L)} \tag{3.102}$$

$b_R = \xi + \sqrt{\cdots}$ and $b_L = \xi - \sqrt{\cdots}$. The factor of $\xi^2 - e^2/4F$ must be positive. In figure 3.6b, this constraint means that E is less than the top of the potential barrier.

The above integral is expressed in terms of complete elliptic integrals:

$$\alpha = \frac{2\sqrt{2mFb_R}}{3\hbar^2}[2\xi E(p) - 2b_L K(p)], \quad p^2 = \frac{b_R - b_L}{b_R} \tag{3.103}$$

This expression is usually evaluated on the computer. The image correction to Fowler-Nordheim tunneling is most important at large values of field F.

3.4 Variational Theory

The variational method is useful for finding the eigenfunction and eigenvalue of the lowest bound state of a Hamiltonian. The lowest bound state is called the *ground state*.