Time evolution of a shifted initial state:

Let $\phi(x,t)$ be the time evolution of an initial function $\phi(x)$ in the field of the harmonic oscillator. For $x_0 \in \mathbb{R}$ define the shifted function by $\phi_{x_0}(x) = \phi(x - x_0)$. Its time evolution is given by

$$\phi_{x_0}(x,t) = \exp\Bigl(\mathrm{i}\,\frac{x_0^2}{4}\,\sin 2t\Bigr)\,\mathrm{e}^{-\mathrm{i}(x_0\,\sin t)\,x}\,\phi(x - x_0\,\cos t, t) \tag{7.81}$$

We can draw some interesting conclusions from this result: Assume that $\phi = \phi_n$ is an eigenstate of the harmonic oscillator. The eigenstates have a trivial time evolution, they always remain centered at the origin, the expectation values of the position x and momentum p are zero for all times. Now, lets shift the eigenfunction to a new position x_0, that is, the new initial state is given by

$$\psi(x,0) = \phi_n(x - x_0). \tag{7.82}$$

Equation (7.81) gives the following result for the time evolution of the translated eigenstate:

$$\psi(x,t) = \exp\Bigl(\mathrm{i}\,\frac{x_0^2}{4}\,\sin 2t - \mathrm{i}E_n t - \mathrm{i}(x_0\,\sin t)\,x\Bigr)\,\phi_n(x - x_0\,\cos t). \tag{7.83}$$

In particular, the position probability density is just given by

$$|\psi(x,t)|^2 = |\phi_n(x - x_0\,\cos t)|^2. \tag{7.84}$$

Time evolution does not change the shape of $|\phi_n|^2$, it just translates the function to its classical position $x_0 \cos t$.

CD 5.19 shows the motion of the initially translated eigenstates ϕ_1, ϕ_2, and their superposition. The evolution of the shifted initial state shows the motion of $\phi_1 + \phi_2$ as in CD 5.5, while the center of the wave packet performs the oscillation $x_0\,\cos t$.

7.5. Motion of Gaussian Wave Packets

7.5.1. Coherent states

Here we apply the results of the previous section to the ground state $\phi_0(x) = \pi^{(-1/4)}\,\exp(-\frac{1}{2}\,x^2)$. Among the eigenstates it is distinguished also by the property that it is optimal with respect to the uncertainty relation

$$\Delta x \Delta p \geq \tfrac{1}{2}. \tag{7.85}$$

The ground state satisfies

$$\Delta x \Delta p = \tfrac{1}{2}. \tag{7.86}$$

For the time evolution of the initially translated ground state we can apply the results obtained in the previous section. We obtain

$$\psi(x,t) = \Bigl(\frac{1}{\pi}\Bigr)^{1/4} \exp\Bigl(\mathrm{i}\,\frac{x_0^2}{4}\,\sin 2t - \mathrm{i}\,\frac{t}{2}\Bigr)\,\exp\Bigl(\mathrm{i}\,p_t\,x - \frac{(x-x_t)^2}{2}\Bigr), \qquad (7.87)$$

with $x_t = x_0\,\cos t$ and $p_t = -x_0\,\sin t$. This is a normalized Gaussian function centered at the average position x_t and with average momentum p_t. Here (x_t, p_t) describes the classical oscillation of a particle with initial position x_0 (in dimensionless units).

CD 5.10 contains movies of a coherent state in position space, momentum space, and in the energy representation. See also Color Plate Fi:HOcoh.

We know already that the function $\psi(x,t)$ is again optimal with respect to the uncertainty relation (see Section 2.8.1). The state $\psi(x,t)$ hence satisfies Eq. (7.86) for all times. The states with minimal uncertainty are called *coherent states*. Their motion is most similar to the oscillation of a particle in classical mechanics. We collect these observations in the following box.

Coherent states of a harmonic oscillator:

The coherent states (states with minimal uncertainty) of a harmonic oscillator are shifted Gaussian functions, initially localized at x_0, which have the shape of the ground state. The maximum of a coherent state always follows the trajectory of a classical-mechanical particle that starts at x_0 with zero initial momentum. The wavelength of the phase always corresponds to the momentum of the classical particle. During its time evolution a coherent state retains its shape (the shape of the ground state) without spreading.

It is a consequence of Exercise 7.10 that the coherent states minimize the difference between the mean energy and the energy of the classical motion of $\langle x(t)\rangle_\psi$ and $\langle p(t)\rangle_\psi$ because the coherent states have minimal uncertainty. One even has the following result: ψ is a coherent state if and only if

$$\tfrac{1}{2}\Bigl(\langle p\rangle_{\psi(t)}^2 + \langle x\rangle_{\psi(t)}^2\Bigr) = \tfrac{1}{2}\langle p^2 + x^2\rangle_{\psi(t)}. \qquad (7.88)$$

7.5.2. Arbitrary Gaussian function

For the harmonic oscillator potential, the Schrödinger equation with the Gaussian initial function

$$\phi(x) = \Bigl(\frac{a}{\pi}\Bigr)^{1/4} \exp\Bigl(-a\,\frac{x^2}{2}\Bigr) \qquad (7.89)$$

has the solution

$$\psi(x,t) = \left(\frac{a}{\pi}\right)^{1/4} \left(\cos t + \mathrm{i}a\sin t\right)^{-1/2} \exp\left(-a(t)\,\frac{x^2}{2}\right), \tag{7.90}$$

where

$$a(t) = \frac{a\,\cos t + \mathrm{i}\,\sin t}{\cos t + \mathrm{i}a\sin t}. \tag{7.91}$$

This result can be obtained using the explicitly known integral kernel of the time evolution. This integral kernel (Mehler's kernel) will be obtained in a later section.

One has to be careful with the definition of the square root

$$\left(\cos t + \mathrm{i}a\sin t\right)^{-1/2} \tag{7.92}$$

in Eq. (7.90). It is necessary to take that branch of the square root that gives a continuous dependence on t. Hence for $a = 1$ this expression will simply become $\exp(-\mathrm{i}t/2)$.

By combining Eq. (7.90) with Eq. (7.81) one can easily find an expression for the time evolution of a translated Gaussian function.

7.6. Harmonic Oscillator in Two and More Dimensions

The wave function of a particle in two dimensions depends on a two-dimensional position variable $\mathbf{x} = (x_1, x_2)$. The Hamiltonian operator for the harmonic oscillator in two dimensions can be written as a sum of one-dimensional Hamiltonians

$$\begin{aligned} H &= -\frac{1}{2}\Delta + \frac{\mathbf{x}\cdot\mathbf{x}}{2} \\ &= -\frac{1}{2}\frac{d^2}{dx_1^2} + \frac{x_1^2}{2} - \frac{1}{2}\frac{d^2}{dx_2^2} + \frac{x_2^2}{2} \\ &= H_{x_1} + H_{x_2}, \end{aligned}$$

where H_{x_i} $(i = 1, 2)$ acts only on the variable x_i. Hence we can make the same observations as in the case of free particles: The Schrödinger equation in two space dimensions can be solved by a product ansatz

$$\psi(\mathbf{x},t) = \psi_1(x_1,t)\,\psi_2(x_2,t), \tag{7.93}$$

where each ψ_i is a solution of the one-dimensional harmonic oscillator equation.

$$\begin{aligned}\mathrm{i}\frac{d}{dt}\psi &= \mathrm{i}\frac{d\psi_1}{dt}\psi_2 + \mathrm{i}\psi_1\frac{d\psi_2}{dt}\\ &= (H_{x_1}\psi_1)\,\psi_2 + \psi_1\,(H_{x_2}\psi_2)\\ &= (H_{x_1} + H_{x_2})\,\psi_1\psi_2 = H\psi.\end{aligned}$$

If $\psi_{n,m}$ is a product of two eigenfunctions ϕ_n and ϕ_m of the one-dimensional Hamiltonian,

$$\psi_{n,m}(\mathbf{x}) = \phi_n(x_1)\,\phi_m(x_2), \tag{7.94}$$

then $\psi_{n,m}(\mathbf{x})$ is an eigenfunction of H belonging to the eigenvalue $E_{n,m}$

$$E_{n,m} = E_m + E_n = m + n + 1, \qquad H\,\psi_{n,m} = E_{n,m}\,\psi_{n,m}. \tag{7.95}$$

Hence it is clear that

$$\psi_{n,m}(\mathbf{x},t) = \psi_{n,m}(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}E_{n,m}t} = \phi_n(x_1,t)\,\phi_m(x_2,t) \tag{7.96}$$

is a solution of the time-dependent Schrödinger equation. This solution is a product of solutions of one-dimensional equations for x_1- and x_2-coordinates, respectively. Of course, this does not mean that *every* solution of the two-dimensional oscillator is a product of one-dimensional solutions. Nevertheless, any solution can be written as an (infinite) linear combination of products as follows.

As a consequence of the fact that the functions ϕ_n form an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$ it can be shown that the product functions $\psi_{n,m}$ form an orthonormal basis in $L^2(\mathbb{R}^2)$ (this is a property of the tensor product of Hilbert spaces). Hence every initial function $\phi \in L^2(\mathbb{R}^2)$ can be expanded as

$$\phi(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(\mathbf{x}), \tag{7.97}$$

and the unique solution of the Schrödinger equation with initial condition $\psi(\mathbf{x},t) = \phi(\mathbf{x})$ is given by

$$\psi(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\mathrm{e}^{-\mathrm{i}(n+m+1)\,t}\,\psi_{n,m}(\mathbf{x}). \tag{7.98}$$

The ground-state energy is $E_0 + E_0 = 1$ and every solution is periodic with period 2π.

EXERCISE 7.12. *Generalize the above considerations to n-dimensions.*

If $\psi(x,t)$ is the time evolution of a one-dimensional harmonic oscillator, then its Fourier transform, the function $\hat{\psi}(k,t)$, is also a solution of the

Schrödinger equation of the harmonic oscillator in one dimension. Hence the wave function in phase space,

$$\Psi(x,k,t) = \psi(x,t)\,\hat{\psi}(k,t) \tag{7.99}$$

is a solution of the two-dimensional oscillator equation.

CD 5.14–5.18 show solutions of the two-dimensional harmonic oscillator. Among the various Gaussian wave packets, the coherent and squeezed states are of particular interest.

7.7. Theory of the Harmonic Oscillator

7.7.1. Supersymmetry

Define the operators

$$A = \frac{1}{\sqrt{2}}\,(x + \mathrm{i}p) = \frac{1}{\sqrt{2}}\Bigl(\frac{d}{dx} + x\Bigr), \tag{7.100}$$

$$A^\dagger = \frac{1}{\sqrt{2}}\,(x - \mathrm{i}p) = \frac{1}{\sqrt{2}}\Bigl(-\frac{d}{dx} + x\Bigr). \tag{7.101}$$

Here $p = -\mathrm{i}d/dx$ is the momentum operator, and x denotes the position operator (the operator of multiplication of $\psi(x)$ by x). For suitable (differentiable) wave functions you can see by a partial integration that

$$\langle\phi, A\psi\rangle = \langle A^\dagger\phi, \psi\rangle. \tag{7.102}$$

Therefore, the operators A and $A^\dagger$ are formally adjoint to each other.

Schwartz space. For the mathematical investigation it is necessary to have a dense domain in the Hilbert space $L^2(\mathbb{R})$, which is invariant under the action of the operators A and $A^\dagger$. Such a domain is the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$. The set $\mathcal{S}$ consists of infinitely differentiable functions. The functions (and all their derivatives) are required to go to zero faster than any inverse power of $|x|$, as $|x|$ tends to infinity. More precisely,

$$\mathcal{S} = \Bigl\{f \in C^\infty(\mathbb{R}) \Bigm| \text{for all integers } k,\ l\text{: } \sup_{x\in\mathbb{R}}\Bigl|x^k \frac{d^l}{dx^l} f(x)\Bigr| < \infty\Bigr\}. \tag{7.103}$$

Typical examples of functions in $\mathcal{S}$ are Gauss functions $\exp(-ax^2)$, and all the oscillator eigenfunctions. Because all finite linear combinations of functions in $\mathcal{S}$ are again contained in $\mathcal{S}$, the set $\mathcal{S}$ is a linear subspace of the Hilbert space $L^2(\mathbb{R})$. Theorem 2.2 in Section 2.7 states that the linear subspace spanned by the functions

$$G_q(x) = \Bigl(\frac{1}{\pi}\Bigr)^{1/4} e^{\mathrm{i}qx} \exp\Bigl(-\frac{x^2}{2}\Bigr) \tag{7.104}$$

3. Wave function in momentum space
4. Wave function in phase space

CD 5.5. **Oscillating state 1+2**

1. Wave function in position space
2. Position probability density
3. Wave function in momentum space
4. Wave function in phase space

CD 5.6. **Superposition of three eigenstates** $(0+1+2)$

1. Wave function in position space
2. Position probability density
3. Wave function in momentum space

CD 5.7. **Linear combination of three eigenstates** $(0+1-2)$

1. Wave function in position space
2. Position probability density

CD 5.8. **Superposition of ten eigenstates**

1. Sum of first ten eigenstates
2. Boltzmann distribution of energies
3. Random phases

CD 5.9. **Eigenfunction expansion**

1. Build a Gaussian function (interactive experiment)
2. Energy representation
3. Energy representation of a shifted state
4. Time evolution in energy space

CD 5.10. **Coherent state (shifted eigenstate)**

1. Wave function in position space
2. Position probability density
3. Wave function in momentum space
4. Energy representation
5. Time period and gauge

CD 5.11. **Squeezed state (widespread initial localization)**

1. Wave function in position space
2. Position probability density
3. Wave function in momentum space

CD 5.12. **Squeezed state (sharper initial localization)**

1. Wave function in position space
2. Position probability density
3. Momentum space

CD 5.13. **Step function**

1. Approximation in position space

2. Position probability density
3. Momentum space

CD 5.14. **Coherent state**

1. Phase space motion
2. Circular motion in two dimensions
3. Elliptic motion in two dimensions

CD 5.15. **Diagonal motion (surface plots)**

1. Coherent state in two dimensions
2. Squeezed state in two dimensions (flat)
3. Squeezed state in two dimensions (sharp)

CD 5.16. **Particle at rest**

1. Centered squeezed state
2. Asymmetric centered state
3. Squeezed state in phase space

CD 5.17. **Phase space motion (contour plots)**

1. Squeezed state in phase space
2. Closer to the center
3. Very flat Gaussian state
4. Sharp Gaussian peak

CD 5.18. **Circular motion (surface plots)**

1. Circular motion of Gaussian
2. Another initial condition
3. Yet another initial condition ("the swimmer")

CD 5.19. **Constant shapes (time evolution of shifted eigenstates)**

1. Translated first eigenstate
2. Second eigenstate
3. First+second eigenstate

CD 5.20. **Gallery of angular momentum eigenstates**

6. Special Systems

CD 6.1. **Free Fall**

1. Dropping a particle
2. Throwing a particle

CD 6.2. **Free fall**

1. Exact solution in a linear potential

CD 6.3. **Free fall in two dimensions**

1. Letting a particle drop vertically
2. Vertical throw of a particle