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New Schrödinger equations for old: Inequivalence of the Darboux and Abraham-Moses constructions

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There exist two methods for generating families of isospectral Hamiltonians: one based on a theorem due to Darboux and the second due to Abraham and Moses based on the Gel'fand-Levitan equation. Both methods start with a general Hamiltonian operator $H=-d^2/dx^2+V(x)$, and generate infinite families of new Hamiltonians all with the same eigenvalue spectrum. The new spectrum corresponds either to the addition of new bound states with specified energy eigenvalues or to the deletion of bound-state eigenvalues. Neither process (addition or deletion) alters the reflection or transmission probabilities, although the amplitudes experience a phase change consistent with Levinson's theorem and the change in the number of bound states. In this paper we show that these two methods of generating families of isospectral Hamiltonians are, in general, inequivalent.

I. INTRODUCTION

There is a widespread misconception among physicists that, for a particle in a one-dimensional confining potential, a complete knowledge of the energy spectrum is sufficient to determine the potential uniquely. For example, most physicists regard equally spaced energy eigenvalues as signaling that the potential is necessarily proportional to x^2, as in the harmonic oscillator. In fact, as an example of a more general procedure, Abraham and Moses[1] (AM) have explicitly constructed families of *anharmonic* potentials which give rise to the harmonic-oscillator energy spectrum. Their procedure, based on the Gel'fand-Levitan equation,[2] when applied to an arbitrary initial Hamiltonian H, allows the construction of continuous families of isospectral[3] Hamiltonians, i.e., new Hamiltonians with the same eigenvalue spectrum as H. There also exists an alternative procedure[3,4] for generating families of isospectral Hamiltonians, which is based on a theorem for second-order linear differential equations discovered over a century ago by Darboux.[5,6] With this procedure, starting with any initial Hamiltonian H with a ground-state energy E_0, one can generate a one-parameter family of new Hamiltonians all of which have the same eigenvalue spectrum as H except for the occurrence of a new ground-state energy E_{-1}. The value of E_{-1} can be selected at will; it is subject to only one constraint: namely, $E_{-1}<E_0$. Similarly, the Darboux construction can be used (as long as the ground state of H is bound) to construct a unique new Hamiltonian which shares the same eigenvalue spectrum as H except that E_0 has been deleted. Obviously these alternate procedures, either inserting a new ground-state eigenvalue or deleting a (bound) ground-state eigenvalue, can be applied iteratively. It should be noted that neither the Darboux nor the AM method is restricted to Hamiltonians with a confining potential. They can be applied to one-dimensional Hamiltonians with completely arbitrary potentials.

This article is the first of a series of papers exploring various procedures for generating families of isospectral Hamiltonians. A subsequent article[7] demonstrates yet a third general method for creating families of Hamiltonians which share a common eigenvalue spectrum. The relations between the three main procedures will be examined in more detail in a third article,[8] using the techniques of isometric operators. Yet more families of isospectral Hamiltonians, generated by iterative combinations of the three basic procedures, will be treated in a fourth paper,[9] which will also summarize the relationships between all the methods considered.

The purpose of this first article of the series is to demonstrate the inequivalence of the AM and Darboux procedures for generating families of isospectral Hamiltonians. For scattering potentials, i.e., potentials which converge to zero faster than x^{-1} as $|x|\to\infty$, the differences between the Darboux and AM constructions become particularly transparent. If one inserts a new (or deletes the old) ground-state eigenvalue, the two procedures generate different families of new Hamiltonians with the same eigenvalue spectrum but differing reflection coefficients and "norming constants." For an arbitrary potential, the Darboux and AM procedures are equivalent only for the following case. Suppose one first deletes the (bound) ground-state eigenvalue E_0 of H and then reintroduces a new ground state with the same energy E_0, using either the Darboux construction or the AM procedure for both steps. The two methods then yield the same one-parameter family of Hamiltonians having the energy eigenvalue spectrum of the original Hamiltonian. Furthermore, the same result could be obtained in one step using the AM technique to "renormalize" the original ground state. This fact was used by Nieto[10] to analyze the relations between supersymmetric partner Hamiltonians, the general Darboux procedure, and the AM technique for "renormalizing" the ground state.

This article is organized as follows. In Sec. II, we summarize the main features of the Darboux construction for both confining and scattering potentials. The corresponding results for the AM procedure are summarized in Sec. III. These two sections establish a uniform notation and a convenient reference for the main part of the paper, making this article essentially self-contained. The relationship between the two procedures is analyzed in Sec. IV. There we demonstrate that the two methods are in general inequivalent when used to add a new or delete the old ground-state eigenvalue, and we also derive the exceptional cases in which the constructions are equivalent. In Sec. IV we also show that the two procedures are always equivalent when they are used to "renormalize" the ground state. Finally, we summarize our conclusions in Sec. V.

Before concluding this Introduction, we briefly survey the physics literature in which the Darboux procedure has been exploited. Variations and special cases of the Darboux procedure have been developed by many authors (often, it would appear, unaware of Darboux's work). Thus the factorization method developed by Schrödinger[11] and extended by Infeld and his collaborators[12] is in its essence a special case of the Darboux procedure. Crum[13] generalized the Darboux construction, and his work stimulated interest in the use of the procedure in the inverse scattering problem. Krein[14] and Faddeev[15] have made significant applications of the method to the three-dimensional inverse scattering problem, assuming spherical symmetry, while Deift and Trubowitz[4] have used it in their analysis of the one-dimensional problem. Also considering three dimensions, Baumgartner, Grosse, and Martin[16] have exploited the method to develop theorems on level ordering in potential models. Applications of the Darboux construction to soliton theory[17] arise through the application of inverse-scattering methods to the solution of the Korteweg–de Vries equation. Supersymmetric quantum mechanics[18] succeeds in combining two essentially isospectral Hamiltonians into a single Schrödinger equation by introducing additional fermionic degrees of freedom. The two supersymmetric partner Hamiltonians are related by a special case of the Darboux construction.[10] Lastly, Adrianov, Borisov, and Ioffe[19] have used the method to delete the ground-state eigenvalue of a one-dimensional system, while Mielnik,[20] using a procedure mathematically equivalent to Darboux's method, has constructed anharmonic oscillator Hamiltonians which are isospectral with that of the harmonic oscillator. Deift[3] provides a very thorough exposition of the Darboux procedure, treating it as a special case of a more general "commutation formula."

II. THE DARBOUX CONSTRUCTION

In this section we summarize the major features of the Darboux construction, which can be used to generate families of isospectral Hamiltonians starting from any nonrelativistic one-dimensional Hamiltonian. Darboux's theorem[5,6] can be stated as follows. Let ψ be the general solution of the Schrödinger equation

$$H\psi(x)\equiv\left[-\frac{d^2\psi}{dx^2}+V(x)\right]\psi(x)=E\psi(x)\ , \tag{1}$$

where E is an arbitrary parameter. Let ϕ be a particular solution[21] of Eq. (1) corresponding to a specific value ϵ of the parameter E. If $E\neq\epsilon$ then

$$\hat{\psi}=\frac{1}{\phi}W(\psi,\phi) \tag{2}$$

is the general solution of the equation

$$\hat{H}\hat{\psi}(x)=E\hat{\psi}(x)\ , \tag{3}$$

where

$$\hat{H}=-\frac{d^2}{dx^2}+\hat{V}(x)\ , \tag{4}$$

$$\hat{V}(x)=V(x)-2\frac{d^2}{dx^2}\ln[\phi(x)]\ , \tag{5}$$

and $W(f,g)\equiv[f(dg/dx)-(df/dx)g]$ is the Wronskian of $f(x)$ and $g(x)$. It is easy to verify these results by direct substitution.

We impose the constraint that $\phi(x)$ be free of all real finite zeros, in order that $\hat{V}(x)$ may be free of singularities. This requirement can be met only if $\epsilon\leq E_0$, where E_0 is the ground-state eigenvalue of H. Initially, we consider $\epsilon<E_0$, postponing consideration of $\epsilon=E_0$ until Sec. II C. This condition on ϵ ensures the existence of a solution of Eq. (1) with $E=\epsilon$ satisfying the boundary condition

$$u(x)\rightarrow 0\quad(x\rightarrow-\infty)\ , \tag{6}$$

and free of zeros for finite real x. The general solution of Eq. (1) with $E=\epsilon$ is then

$$\phi(x)=u(x)\left[\alpha+\int_x^\infty dy\,[u(y)]^{-2}\right]\ . \tag{7}$$

The condition that $\phi(x)$ has no real zeros (finite or infinite) is met by requiring

$$0<\alpha<\infty\ . \tag{8}$$

A. Insertion of a new ground state

Let $\phi(x)$ be the zero-free solution of Eq. (1) (with $E=\epsilon$) defined by Eq. (7), where ϵ is smaller than the ground-state energy E_0 of H. The Darboux theorem leads to the following remarkable results: the eigenvalue spectrum of $\hat{H}$, given by Eq. (4), is identical with that of H except for the addition of a new ground-state eigenvalue $E_{-1}\equiv\epsilon$. Furthermore, the normalized ground-state eigenfunction of $\hat{H}$ is given by

$$\hat{\psi}_{-1}(x)=\alpha^{1/2}\frac{1}{\phi(x)}\ , \tag{9}$$

while the normalized excited state eigenfunctions are

$$\hat{\psi}_n(x)=-(E_n-\epsilon)^{-1/2}\phi(x)\frac{d}{dx}\left[\frac{1}{\phi(x)}\psi_n(x)\right]\quad(n\geq 0)\ . \tag{10}$$

The condition that $\phi(x)$ has no real zeros, finite or infinite, is necessary in order that $\hat{\psi}_{-1}$ be normalizable. The eigenfunctions $\hat{\psi}_n(x)$, $n \geq -1$, defined by Eqs. (9) and (10) are normalized to unity and (when supplemented by the scattering solutions) can be shown to form a complete set.

B. Scattering

In this section, we assume that the potential $V(x)$ decreases to zero sufficiently rapidly as $x \to \pm\infty$ for there to exist scattering states with energy k^2 and eigenfunctions $\psi(k,x)$ having the asymptotic behavior

$$\psi(k,x) \to e^{ikx} + R(k)e^{-ikx} \quad (x \to -\infty) , \tag{11a}$$

$$\psi(k,x) \to T(k)e^{ikx} \quad (x \to +\infty) . \tag{11b}$$

We now set $\epsilon \equiv E_{-1} = -\kappa^2 < E_0$, and introduce a new ground-state with energy ϵ by the procedure described in Sec. II A. Equation (2) may be used to construct the scattering solutions of Eq. (3) from those of Eq. (1). The reflection and transmission amplitudes $\hat{R}(k)$ and $\hat{T}(k)$ for the new Hamiltonian $\hat{H}$ may then be expressed in terms of the corresponding amplitudes for H by

$$\hat{R}(k) = -e^{i\delta(k)}R(k), \quad \hat{T}(k) = e^{i\delta(k)}T(k) , \tag{12}$$

where

$$e^{i\delta(k)} = \frac{-\kappa + ik}{+\kappa + ik} . \tag{13}$$

It follows that the reflection and transmission probabilities produced by the new potential are exactly the same as those produced by the original potential. However, there is a change in the scattering phase shift of amount $\delta(k)$. We note that $\delta(0) - \delta(\infty) = \pi$. This is exactly what we should expect according to Levinson's theorem,[22] since we have added one extra bound state.

If there are bound states, the potential is not uniquely determined from $R(k)$ together with the bound-state energy eigenvalues, even if the potential converges to zero faster than x^{-2} as $|x| \to \infty$. The necessary additional data are the "norming constants"[4] for the bound states. If $\psi_n(x), n \geq 0$, is a bound-state eigenfunction of H, normalized to unity, corresponding to the eigenvalue $E_n = -\kappa_n{}^2$, then the norming constant C_n of the state is defined to be

$$C_n = \left[\lim_{x \to \infty} [\exp(\kappa_n x)\psi_n(x)] \right]^2 . \tag{14}$$

The norming constant $\hat{C}_n$ for $\hat{\psi}_n(x)$, $n \geq 0$, can be shown to be

$$\hat{C}_n = \left[\frac{\kappa + \kappa_n}{\kappa - \kappa_n} \right] C_n . \tag{15}$$

To find the norming constant $\hat{C}_{-1}$ for the new ground state with eigenfunction $\hat{\psi}_{-1}(x)$, we first define σ by

$$\sigma = \lim_{x \to \infty} [e^{-\kappa x}u(x)] . \tag{16}$$

The desired norming constant is then

$$\hat{C}_{-1} = \alpha^{-1} |\sigma|^{-2} . \tag{17}$$

We see that specifying the norming constant of the new ground state removes the ambiguity in the potential represented by the freedom to choose the value of α.

C. Deleting the old ground state

In Sec. II A we showed that if $\phi(x)$ was a solution of $H\phi = \epsilon\phi$ with $\epsilon < E_0$, where E_0 is the ground-state eigenvalue of H, then the new Hamiltonian $\hat{H}$, Eq. (4), has the same eigenvalue spectrum as H except for the addition of a new ground-state eigenvalue $E_{-1} = \epsilon$. In this section we remark that if the ground state of H is bound, we can construct a new Hamiltonian $\hat{H}$ whose eigenvalue spectrum differs from that of H only in that the ground-state eigenvalue E_0 has been deleted. This is achieved by setting $E = E_0$ in Eq. (1) while keeping the definitions of $\hat{H}(x)$ and $\hat{V}$ in Eqs. (4) and (5). In this case, consistency requirements force the choice $\phi(x) = \psi_0(x)$, so that the Darboux construction yields a unique new Hamiltonian

$$\hat{H} = H - 2\frac{d^2}{dx^2}[\ln\psi_0(x)] . \tag{18}$$

If this procedure is applied to any member of the one-parameter family of Hamiltonians constructed in Sec. II A, then we recover the original Hamiltonian H. In this sense, then, the present procedure is the converse of that treated in Sec. II A.

D. Iterations galore

The Darboux procedure can be iterated in a variety of ways, either inserting or removing a ground-state eigenvalue at each stage.[23] Here we consider only the removal of the ground state followed by reinsertion of a new state with the original ground-state energy.

The first step proceeds as in Sec. II C to create a new Hamiltonian $\hat{H}$ given by Eq. (18). This Hamiltonian is uniquely determined, and has eigenvalues E_n, $n \geq 1$. For the second step, one first recognizes that the most general zero-free solution of $\hat{H}\hat{\phi}(x) = E_0\hat{\phi}(x)$ is, up to a normalization constant,

$$\hat{\phi}(x) = \left[\frac{1}{\psi_0(x)} \right] \left[1 + \gamma \int_{-\infty}^{x} dy [\psi_0(y)]^2 \right] . \tag{19}$$

The final Hamiltonian $\tilde{H} = \hat{H} - 2(d^2/dx^2)\ln\hat{\phi}(x)$, is isospectral with the original Hamiltonian H and is given by $\tilde{H} = (-d^2/dx^2) + \tilde{V}(x)$, where

$$\tilde{V}(x) = V(x) - 2\frac{d^2}{dx^2} \left[\ln \left[1 + \gamma \int_{-\infty}^{x} dy [\psi_0(y)]^2 \right] \right] . \tag{20}$$

III. THE ABRAHAM-MOSES RESULTS

In this section we summarize those results of Abraham and Moses[1] which are most relevant to our work, after first translating them into the notation of the present paper. Specifically, we display the expressions these authors obtain for the new Hamiltonian and its eigenfunctions for three special cases: the introduction of a new (bound) ground state at a lower energy than that of the original

Hamiltonian, the deletion of the original ground state, and "renormalization" of the original ground state. For the first of these, the insertion of a new bound state, we also state the effect of the procedure on the reflection and transmission amplitudes and on the norming constants of bound states.

For the specialization of the AM method to each of these processes, one starts with a solution $u(x)$ of Eq. (1) (with a particular value ϵ chosen for E) which satisfies the boundary condition Eq. (6). Of central importance in the AM procedure is the kernel $K(x,y)$ which, for the three special cases considered here, is given by

$$K(x,y) = -\frac{\gamma u(x) u(y)}{1+\gamma I(x)} , \tag{21}$$

where γ is a constant and

$$I(x) = \int_{-\infty}^{x} dy\,[u(y)]^2 . \tag{22}$$

The permissible range of values for the constant γ depends on the specific process chosen, in a manner to be explained later.

In all three of the cases considered here, the new Hamiltonian generated by the AM method may be written as $\hat{H}$ in Eq. (4), but with $\hat{V}(x)$ given by

$$\begin{aligned}\hat{V}(x) &= V(x) + 2\frac{d}{dx}K(x,x)\\ &= V(x) - 2\frac{d^2}{dx^2}\ln[1+\gamma I(x)] , \end{aligned}\tag{23}$$

instead of by Eq. (5). If $\psi(x)$ is any solution of Eq. (1) for arbitrary E, then

$$\hat{\psi}(x) = \psi(x) + \int_{-\infty}^{x} dy\, K(x,y)\psi(y) \tag{24}$$

is a solution of Eq. (3) with $\hat{H}$ given by Eqs. (4) and (23).

To insert a new (bound) ground state with energy $E_{-1} < E_0$, we choose $\epsilon = E_{-1}$ and $\gamma > 0$. The new potential $\hat{V}$ is given by Eq. (23). The normalized eigenfunction for the new ground state is

$$\hat{\psi}_{-1}(x) = \gamma^{1/2}\frac{u(x)}{1+\gamma I(x)} \quad (\gamma > 0) . \tag{25}$$

If $V(x)$ is a scattering potential, then the norming constant of the new ground state is

$$\hat{C}_{-1} = \frac{4\kappa^2}{\gamma\sigma^2} . \tag{26}$$

For other bound states, the eigenfunctions $\hat{\psi}_n(x)$ are determined from $\psi_n(x)$ using Eq. (24) together with Eqs. (21) and (22), and are already normalized to unity. However, if $V(x)$ is a scattering potential, the norming constant is changed to $\hat{C}_n$ given by

$$\hat{C}_n = \left[\frac{\kappa+\kappa_n}{\kappa-\kappa_n}\right]^2 C_n . \tag{27}$$

For scattering solutions, it is obvious that

$$\hat{R}(k) = R(k) , \tag{28}$$

while the new transmission amplitude is

$$\hat{T}(k) = \left[\frac{-\kappa+ik}{+\kappa+ik}\right] T(k) . \tag{29}$$

To remove the (bound) ground state of the original Hamiltonian, we choose $\epsilon = E_0$, so that (apart from normalization) $u(x) \equiv \psi_0(x)$. In this case, one must set $\gamma = -1$. From Eq. (23) together with the normalization condition for $\psi_0(x)$, it follows that

$$\hat{V}(x) = V(x) - 2\frac{d^2}{dx^2}\ln\left[\int_x^{\infty} dy\,[\psi_0(y)]^2\right] . \tag{30}$$

The normalized bound-state eigenfunctions of $\hat{H}$ are $\hat{\psi}_n(x)$, $n \geq 1$, given by Eq. (24) with $u(x)$ in Eqs. (21) and (22) set equal to $\psi_0(x)$.

Finally, to "renormalize" the original (bound) ground state, we set $\epsilon = E_0$, $u(x) \equiv \psi_0(x)$, and choose $\gamma > -1$. The new potential is again given by Eq. (23). For $n \geq 1$, the normalized eigenfunctions of $\hat{H}$ are given by Eq. (24) with $u(x) = \psi_0(x)$ in Eqs. (21) and (22), while the new normalized ground-state eigenfunction is given by

$$\hat{\psi}_0(x) = (\gamma+1)^{1/2}\frac{\psi_0(x)}{1+\gamma I(x)} . \tag{31}$$

IV. COMPARISON WITH THE ABRAHAM-MOSES TECHNIQUE

In this section we show that except for a few special cases the Darboux construction leads to families of new Hamiltonians which are inequivalent to those generated by the method of Abraham and Moses.[1]

In Sec. III we reproduced results obtained by the AM technique for three special cases: namely, insertion of a new ground state, deletion of the original ground state, and renormalization of the original ground state. We also presented the effects of the AM construction on the reflection and transmission amplitudes and on the bound-state norming constants.

First, let us compare the techniques for inserting a new ground state with energy $E_{-1} < E_0$. If the two techniques are to agree, then $\hat{V}(x)$ as given by Eq. (5) together with Eq. (7) must be identical with $\hat{V}(x)$ given by Eq. (23). This is possible only if

$$\begin{aligned}1+\gamma\int_{-\infty}^{x} dy\,[u(y)]^2 &\\ &= e^{A+Bx}u(x)\left[\alpha + \int_x^{\infty} dy\,[u(x)]^{-2}\right] ,\end{aligned}\tag{32}$$

where A and B are integration constants. Also, $\hat{\psi}_{-1}(x)$ given by Eq. (8) must be identical with $\hat{\psi}_{-1}(x)$ given by Eq. (25). This yields

$$\begin{aligned}1+\gamma\int_{-\infty}^{x} dy\,[u(y)]^2 &\\ &= \left[\frac{\gamma}{\alpha}\right]^{1/2}[u(x)]^2\left[\alpha + \int_x^{\infty} dy\,[u(y)]^{-2}\right] .\end{aligned}\tag{33}$$

Comparison of Eqs. (32) and (33) shows that, for all values of x,

$$u(x) = \left[\frac{\alpha}{\gamma}\right]^{1/2} e^{A+Bx} . \tag{34}$$

In this equation, B must be chosen to be positive in order to satisfy the boundary condition Eq. (6). When Eq. (34) is substituted into Eq. (1), we find that $V(x)$ must be a constant so that the original quantum system with Hamiltonian H is essentially a free particle. For simplicity, we may choose $V(x)\equiv 0$, in which case $\epsilon \equiv E_{-1} = -B^2$. If $u(x)$, given by Eq. (34), is substituted back into either Eq. (32) or Eq. (33), then consistency requires that α and γ be related by

$$\frac{\gamma}{\alpha} = 4B^2 \ . \tag{35}$$

Hence the Darboux construction and the AM technique can yield the same results when used to insert a new ground state only if one starts from the free-particle Hamiltonian. In fact, for this special case, one can show that the potential $\hat{V}(x)$ for the new system is a particular case of the modified Pöschl-Teller potential[24]

$$V(x) = -\frac{\lambda(\lambda-1)B^2}{\cosh^2[B(x-a)]} \ , \tag{36}$$

with $\lambda = 2$ and a an arbitrary real constant.

The inequivalence of the Darboux and AM methods becomes more transparent when $V(x)$ is a scattering potential. Comparison of Eqs. (12) and (13) with Eqs. (28) and (29), and Eq. (15) with Eq. (27) shows that while the two procedures have the same effect on the transmission amplitude, they have quite different effects on both the reflection amplitude and on the norming constants of any bound states of the original Hamiltonian. Hence the only initial Hamiltonians H for which they could be expected to produce the same family of new potentials $\hat{H}$ must be ones which have no bound states and which produce a vanishing reflection coefficient. These conditions uniquely determine the class of Hamiltonians with constant potentials, i.e., free-particle Hamiltonians.

We next compare the two methods when they are used to eliminate a bound ground state. If the two procedures are to agree, then $\hat{H}$ given by Eq. (18) must be identical with $\hat{H}$ given by Eq. (4) with $\hat{V}(x)$ given by Eq. (30). Hence we conclude that

$$\int_x^{\infty} dy\, [\psi_0(y)]^2 = e^{-A-Bx}\psi_0(x) \ , \tag{37}$$

where as before A and B are integration constants. After differentiating Eq. (37) and rearranging terms, we find

$$-\frac{d\psi_0(x)}{dx} + B\psi_0(x) = e^{A+Bx}[\psi_0(x)]^2 \ . \tag{38}$$

The general solution of this equation is

$$\frac{1}{\psi_0(x)} = \frac{1}{2B}(e^{A+Bx} + C\,e^{-Bx}) \ , \tag{39}$$

where C is an integration constant which must be positive in order that $\psi_0(x)$ be free of singularities. This result can be rewritten as

$$\psi_0(x) = \frac{K}{\cosh[B(x-a)]} \ , \tag{40}$$

where

$$K = BC^{-1/2}e^{-A/2}, \quad e^{Ba} = C^{1/2}e^{-A/2} \ . \tag{41}$$

It is easily verified that Eqs. (40) and (41) are consistent with Eq. (37). The condition that $\psi_0(x)$ be normalized to unity yields

$$K = (\tfrac{1}{2}B)^{1/2} \ . \tag{42}$$

The potential $V(x)$ which gives rise to the ground-state eigenfunction $\psi_0(x)$, given in Eq. (40), can be computed from $V(x) = E_0 + [d^2\psi_0(x)/dx^2]/\psi_0(x)$, and is given by

$$V(x) - E_0 = B^2 - 2\left\{\frac{B}{\cosh[B(x-a)]}\right\}^2 \ . \tag{43}$$

This potential is a particular case of the modified Pöschl-Teller potential, Eq. (36). Indeed it is precisely the new potential $\hat{V}(x)$ which one obtains when one creates a bound ground state starting from the free-particle Hamiltonian. In summary, only if $V(x)$ is given by Eq. (43) do the Darboux and the AM methods for removing a ground state yield the same results.

Finally, we consider the results of first removing the (bound) ground state and then reintroducing a new ground state with the same energy eigenvalue, as was discussed in Sec. II D. The final potential obtained in this way by the Darboux construction is given in Eq. (20). Comparison of that equation with Eqs. (23) and (22) shows that this particular two-stage process using the Darboux construction is always equivalent to renormalization of the ground-state eigenfunction by the AM technique. This equivalence has been noted and used by Nieto[10] in his analysis of the relation between the general Darboux procedure and the special case of supersymmetric partner Hamiltonians.

V. CONCLUSIONS

In this article we have compared two procedures for constructing families of isospectral Hamiltonians, and shown them to be inequivalent except for a few special cases. This result is of physical importance since physicists frequently face the problem of determining a potential from such data as the energy eigenvalues of a system. What minimum data is necessary to uniquely determine the potential? Inverse-scattering theory[4,25] provides one class of potentials for which one knows what input data suffices to determine a unique one-dimensional potential. This is the subclass of scattering potentials which decrease to zero faster than x^{-2} for large values of $|x|$. For this restricted class the potential is uniquely determined from the reflection coefficient together with the bound-state energies and norming constants. However, uniqueness is lost[26] for potentials which converge to zero no faster than x^{-2} as $|x| \to \infty$. For confining potentials we are unaware of any theorems which specify the minimum input data, over and beyond the energy eigenvalues, sufficient to determine a unique potential. In the absence of any suitable selection criteria, we expect there to be a host of inequivalent procedures giving rise to different families of Hamiltonians yet all sharing a common eigenvalue spectrum. Indeed, one of us has recently discovered yet a third inequivalent procedure for generating infinite families of isospectral Hamiltonians,[7] and has shown that yet more families can be created by mixing the three known procedures in an iterative manner.[9]

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[1]P. B. Abraham and H. E. Moses, Phys. Rev. A **22**, 1333 (1980). Hereafter, this paper will be referred to as AM.

[2]I. M. Gel'fand and B. M. Levitan, Am. Math. Soc. Transl. **1**, 253 (1951).

[3]P. A. Deift, Duke Math. J. **45**, 267 (1978).

[4]P. A. Deift and E. Trubowitz, Commun. Pure Appl. Math. **32**, 121 (1979).

[5]G. Darboux, C. R. Acad. Sci. (Paris) **94**, 1456 (1882).

[6]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956), p. 132.

[7]D. L. Pursey, Phys. Rev. D (to be published).

[8]D. L. Pursey, report, 1985 (unpublished).

[9]D. L. Pursey, report, 1985 (unpublished).

[10]M. M. Nieto, Phys. Lett. **145B**, 208 (1984).

[11]E. Schrödinger, Proc. R. Irish Acad. **A46**, 9 (1940); **A46**, 183 (1940); **A47**, 53 (1941).

[12]L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951), and references cited therein.

[13]M. M. Crum, Quart. J. Math. **6**, 121 (1955).

[14]M. G. Krein, Dok. Akad. Nauk SSSR **113**, 970 (1957).

[15]L. D. Faddeev, J. Math. Phys. **4**, 72 (1963).

[16]B. Baumgartner, H. Grosse, and A. Martin, Nucl. Phys. **B254**, 528 (1985).

[17]See, for example, G. L. Lamb, *Elements of Soliton Theory* (Wiley, New York, 1980), Sec. 2.6.

[18]E. Witten, Nucl. Phys. **B188**, 513 (1981). For recent applications and extensive citations of other work, see V. Kostelecký and M. M. Nieto, Phys. Rev. Lett. **53**, 2285 (1984); T. D. Imbo and U. P. Sukhatme, *ibid.* **54**, 2184 (1985).

[19]A. A. Adrianov, N. V. Borisov, and M. V. Ioffe, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 78 (1984) [JETP Lett. **39**, 93 (1984)].

[20]B. Mielnik, J. Math. Phys. **25**, 3387 (1984).

[21]Henceforth in this section, functions which are solutions of Eq. (1) but which do not necessarily satisfy the boundary conditions for a physically meaningful wave function will be denoted by $\phi(x)$. Bound-state eigenfunctions will be denoted by $\psi_n(x)$, with corresponding eigenvalues E_n, and are ordered so that $E_{n+1} > E_n$. Bound-state eigenfunctions will always be chosen to be real and to be normalized to unity. Scattering solutions corresponding to energy $E = k^2$ will be denoted by $\psi(k,x)$.

[22]N. Levinson, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **25**, No. 9 (1949); R. G. Newton, *Scattering Theory of Waves and Particles,* 2nd ed. (Springer, New York, 1982), Sec. 12.1.3.

[23]In the context of a particle confined to a finite interval of the x axis, Crum (Ref. 13) has generalized the Darboux procedure to permit the insertion of arbitrarily many new bound states in a single step. His approach is easily adapted to the infinite-range problem considered in this article.

[24]The modified Pöschl-Teller potential is also known as the Eckart potential and the Rosen-Morse potential. The relevant articles are C. Eckart, Phys. Rev. **35**, 1303 (1930); N. Rosen and P. M. Morse, *ibid.* **42**, 210 (1932); G. Pöschl and E. Teller, Z. Phys. **83**, 143 (1933).

[25]Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963); K. Chadan and P. C. Sabatier, *Inverse Problems in Scattering Theory* (Springer, New York, 1980); R. G. Newton, *Scattering Theory of Waves and Particles* (Ref. 22), Chap. 22.

[26]P. B. Abraham, B. DeFacio, and H. E. Moses, Phys. Rev. Lett. **46**, 1657 (1981).