A NLO Calculation of pQCD: Total Cross Section of *W* Boson at Hadron Colliders

C.-P. Yuan
Michigan State University

July 17, 2007 @ AS

Goal: Learn how to carry out a next-to-leading order QCD calculation in which there are typically collinear and soft singularities (in addition to ultraviolet singularity), needed to be cancel to yield finite experimental observables.

Outline

1. Parton Model

⇒ Born Cross Section

2. Factorization Theorem

⇒ How to organize a NLO calculation of pQCD

3. Feynman rules and Feynman diagrams

⇒ "Cut diagram" notation

4. Immediate Problems (Singularities)

⇒ Dimensional Regularization

5. Virtual Corrections

6. Real Emission Contribuation

7. Perturbative Parton Distribution Functions

8. Summary of NLO $[O(\alpha_s)]$ Corrections

Appendices:

$A.$ γ**-matrices in** n **dimensions**

$B.$ **Some integrals and "special functions"**

$C.$ **Angular integrals in** n **dimensions**

$D.$ **Two-particle phase space in** n **dimensions**

$E.$ **Explicit Calculations**

(Typesetting: prepared by Qing-Hong Cao at MSU.)

A few references can be found in

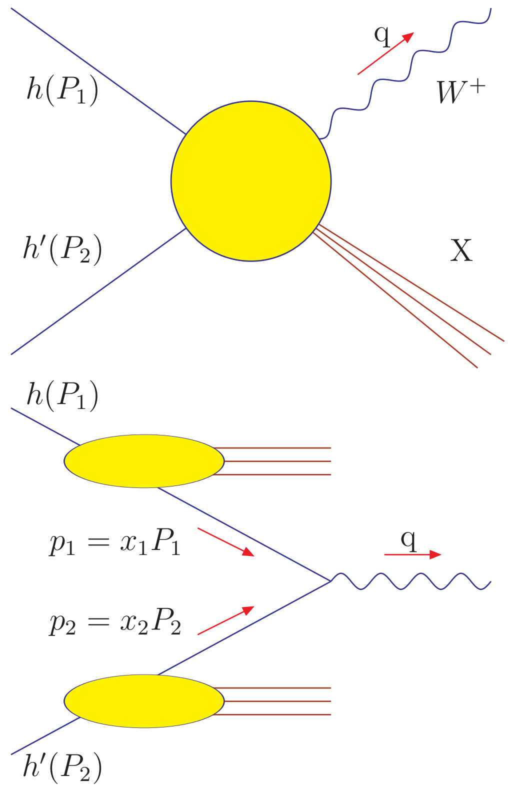
"Handbook of pQCD"

on CTEQ website

http://www.phys.psu.edu/$\sim$ cteq/

Parton Model

$$\sigma_{hh'\to W^+X} =$$



$$=$$

$$\sigma_{hh'\to W^+X} = \sum_{f,f'=q,\bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1)\, \hat{\sigma}_{ff'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

Partonic "Born"
Cross Section of $f\overline{f}' \to W^+$

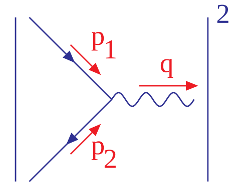
The probablility of finding a "parton" f with fraction x_1 of the hadron h momentum

Born Cross Section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3\, 2q_0} (2\pi)^4\, \delta^4\,(p_1 + p_2 - q) \cdot \overline{|\mathcal{M}|^2}$$

where

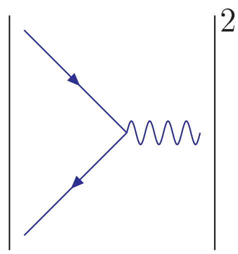
$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{3}\cdot\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)}_{\text{average color and spin}} \sum_{\substack{spin\\ color}}$$



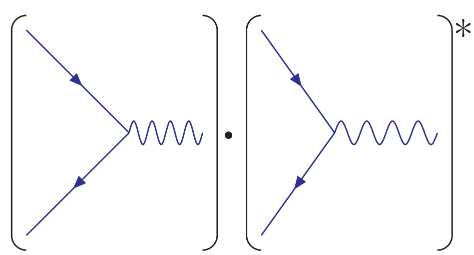
$$\left[\ \text{Or,}\quad -i\mathcal{M} = \bar{v}\,(p_2)\,\frac{ig_w}{\sqrt{2}}\gamma_\mu \frac{1}{2}\,(1-\gamma_5)\,u\,(p_1)\ \right]$$

"Cut-diagram" notation

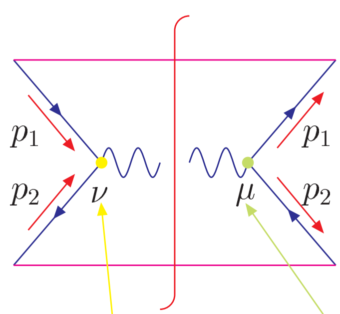
Σ



$= \ \Sigma$



$=$



$\frac{ig_w}{\sqrt{2}}\gamma_\nu P_L$ $\qquad -\frac{ig_w}{\sqrt{2}}\gamma_\mu P_L \qquad P_L \equiv \frac{1}{2}(1-\gamma_5)$

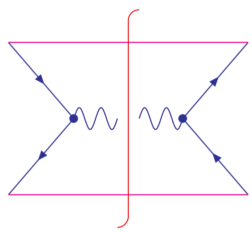
$$= \left(\frac{g_W}{\sqrt{2}}\right)^2 Tr\,[\not{p}_1\gamma_\mu P_L\ \not{p}_2\gamma_\nu P_L] \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \cdot Tr\, I_{3\times 3}$$

Doesn't contribute for $m_q = 0$, due to Ward identity

Color

$$
\begin{aligned}
& Tr\left[\not p_1 \gamma_\mu P_L \not p_2 \gamma^\mu P_L\right](-1) \\
&= Tr\left[\not p_1 \gamma_\mu \not p_2 \gamma^\mu P_L\right](-1) && P_L P_L = P_L = \frac{1}{2}(1-\gamma_5) \\
&= (-2)\, Tr\left[\not p_1 \not p_2 P_L\right](-1) && \gamma_\mu \not p_2 \gamma^\mu = -2 \not p_2 \\
&= (-2)\cdot\frac{1}{2}\cdot 4\,(p_1 \cdot p_2)(-1) && Tr\left(\not p_1 \not p_2\right) = 4\,(p_1\cdot p_2) \\
&= +2\hat{s} && Tr\left(\not p_1 \not p_2 \gamma_5\right) = 0
\end{aligned}
$$

$$
Tr\left[I_{3\times 3}\right] = 3 \qquad \left(\hat{s} \equiv (p_1+p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0\right)
$$

$\Rightarrow$  $= \left(\dfrac{g_w}{\sqrt{2}}\right)^2 \cdot (+2\hat{s})\,(3) = 3\, g_w^2\, \hat{s}$

$$
\begin{aligned}
\int \frac{d^3 q}{2q_0}\delta^4\left(p_1+p_2-q\right) &= \int d^4 q\, \delta^4\left(p_1+p_2-q\right)\delta^+\left(q^2-M^2\right) \\
&= \delta\left(q^2-M^2\right)
\end{aligned}
$$

where M is the mass of W-boson.

Thus,

$$
\begin{aligned}
\hat{\sigma}_{q\bar{q}'} &= \frac{1}{2\hat{s}}\,(2\pi)\cdot\delta\left(\hat{s}-M^2\right)\cdot\left(\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\cdot g_w^2 \hat{s} \\
&= \frac{\pi}{12} g_w^2 \delta\left(\hat{s}-M^2\right) \\
&= \frac{\pi}{12\hat{s}} g_w^2 \delta\left(1-\hat{\tau}\right)
\end{aligned}
$$

$$
\left(\begin{array}{c} \hat{\tau} = M^2/\hat{s}\,,\ \hat{s} = x_1 x_2 S \ \text{ for } \\ S = \left(P_1+P_2\right)^2 \text{ and } P_1^2 = P_2^2 = 0 \end{array}\right)
$$

Factorization Theorem

$$\sigma_{hh'} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/h}\left(x, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/h'}\left(x_2, Q^2\right)$$

Nonperturbative, but universal, hence, measurable

IRS, Calculable in pQCD

Procedure:

(1) Compute σ_{kl} in pQCD with k, l partons

(not h, h' hadron)

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/k}\left(x_1, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/l}\left(x_2, Q^2\right)$$

(2) Compute $\phi_{i/k}, \phi_{j/l}$ in pQCD

(3) Extract H_{ij} in pQCD

H_{ij} IRS $\Rightarrow$ H_{ij} indepent of k, l

$\Rightarrow$ same H_{ij} with $(k \to h, l \to h')$

(4) Use H_{ij} in the above equation with $\phi_{i/h}, \phi_{j/h'}$

Extracting H_{ij} in pQCD

- Expansions in α_s:

$$\sigma_{kl} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sigma_{kl}^{(n)} \qquad \alpha_s = \frac{g^2}{4\pi}$$

$$H_{ij} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n H_{ij}^{(n)}$$

$$\phi_{i/k}(x) = \delta_{ik}\delta(1-x) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \phi_{i/k}^{(n)}$$

$\uparrow$ $\phi_{i/k}^{(0)}$ ($\alpha_s = 0 \Rightarrow$ Parton k " stays itself ")

- Consequences:

$$H_{ij}^{(0)} = \sigma_{ij}^{(0)} = \text{"Born"}$$

suppress "^" from now on

$$H_{ij}^{(1)} = \sigma_{ij}^{(1)} - \left[\ \sigma_{il}^{(0)} \phi_{l/j}^{(1)} + \phi_{k/i}^{(1)} \sigma_{kj}^{(0)}\ \right]$$

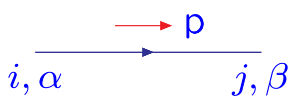
Computed from Feynman diagrams (process dependent) — for $\sigma_{ij}^{(1)}$, $\sigma_{il}^{(0)}$

Computed from the definition of perturbative parton distribution function (process independent, scheme dependent) — for $\phi_{l/j}^{(1)}$, $\phi_{k/i}^{(1)}$

Feynman Rules

- Quark Propagator

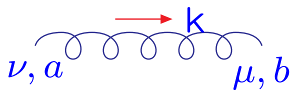
Take m=0 in
our calculation



$$\frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}\delta_{ij}$$

(i,j=1,2,3)

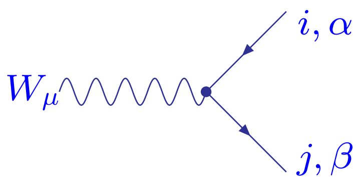
- Gluon Propagator



$$\frac{i(-g_{\mu\nu})}{k^2+i\epsilon}\delta_{ab}$$

(a,b=1,2...,8)

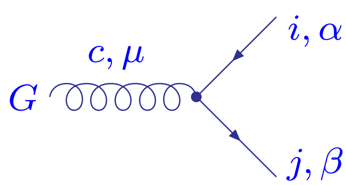
- Quark-W Vertex



$$i\frac{g_W}{\sqrt{2}}(\gamma_\mu)_{\beta\alpha}\frac{(1-\gamma_5)}{2}\delta_{ij}$$

$g_w = \frac{e}{\sin\theta_w}$, weak coupling

- Quark-Gluon Vertex



$$-ig\,(t_c)_{ji}\,(\gamma_\mu)_{\beta\alpha}$$

t_c is the $SU(N)_{N\times N}$ generator

- Quark Color Generators

$$[t_a, t_b] = i f_{abc} t_c$$

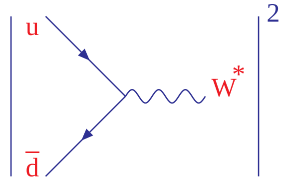
$$\sum_c t_c^2 = C_F\, I_{N\times N}$$

$$Tr(\sum_c t_c^2) = N\, C_F$$

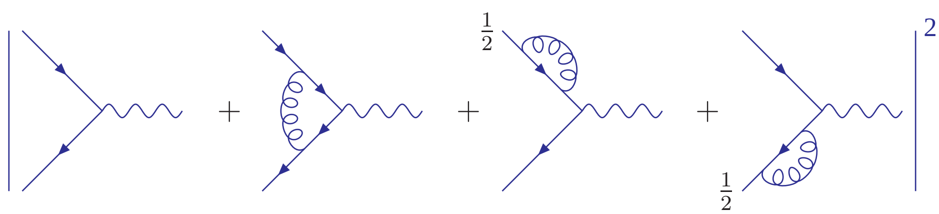
$$C_F = \frac{N^2-1}{2N} = \frac{4}{3}\ ,\ (N=3)$$

Feynman Diagrams

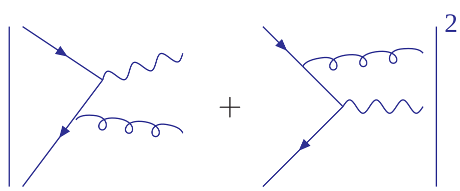
- Born level $\alpha_s^{(0)}$ $(q\overline{q'})_{Born}$



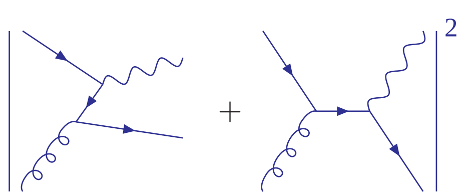
- NLO: $(\alpha_s^{(1)})$ virtual corrections $(q\overline{q'})_{virt}$



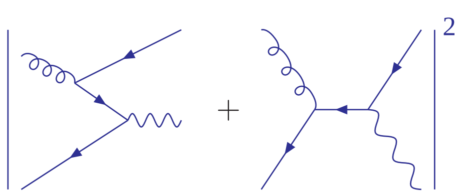
- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(q\overline{q'})_{real}$



- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(qG)_{real}$

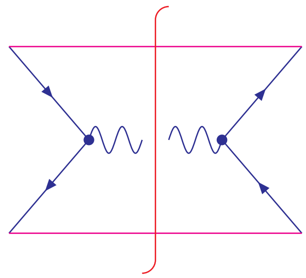


- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(G\overline{q'})_{real}$

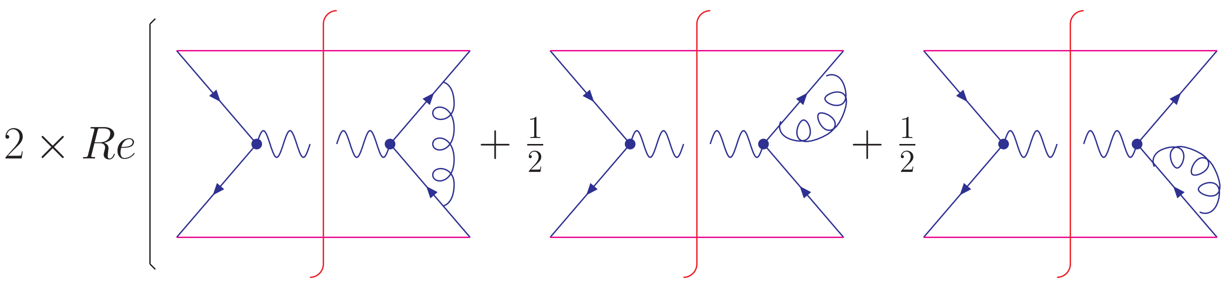


In "Cut-diagram" notation

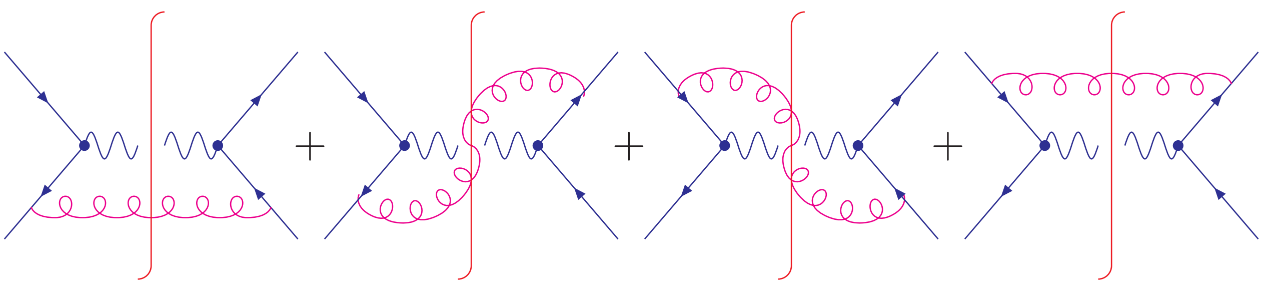
- $(q\overline{q'})_{Born}$



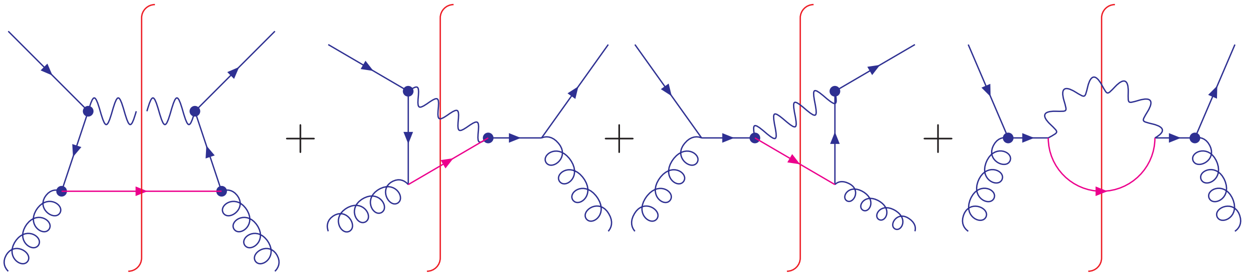
- $(q\overline{q'})_{virt}$



- $(q\overline{q'})_{real}$



- $(qG)_{real}$

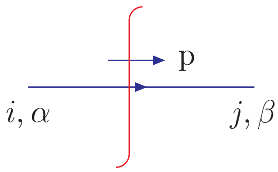


- $(G\overline{q'})_{real}$

Same as $(qG)_{real}$ after replacing q by $\overline{q'}$.

Feynman rules for cut-diagrams

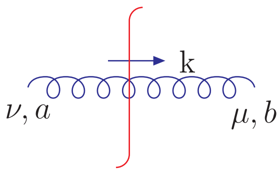
- quark line



$$(2\pi)\delta^+(p^2-m^2)(\not p+m)_{\beta\alpha}\delta_{ij}$$

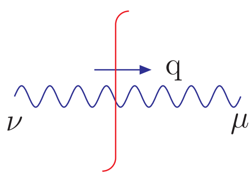
$\delta(p^2-m^2)\theta(p_0)$

- gluon line



$$(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}$$

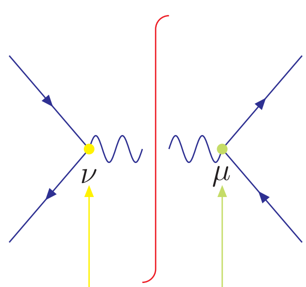
- W-boson line



$$(2\pi)\delta^+(q^2-M^2)(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M^2})$$

Doesn't contribute for $m_f = 0$ because of Ward identity

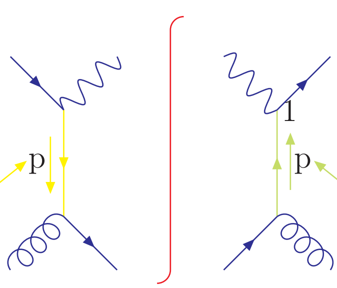
-



$i\frac{g_W}{\sqrt{2}}\gamma_\nu\frac{1}{2}(1-\gamma_5)$

$-i\frac{g_W}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma_5)$

-

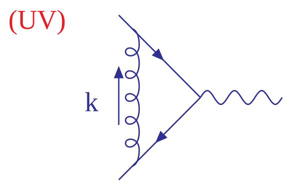


$\frac{i(\not p+m)}{p^2-m^2+i\epsilon}$

$\frac{-i(\not p+m)}{p^2-m^2-i\epsilon}$

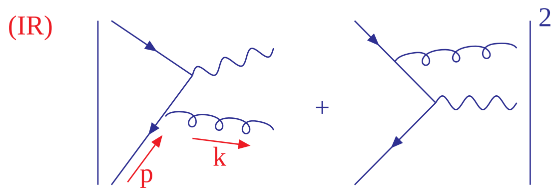
Immediate problems (Singularities)

- Ultraviolet singularity



$$\sim \int^{\infty} d^4k \frac{\not{k}\ \not{k}}{(k^2)\,(k^2)\,(k^2)} \rightarrow \infty$$

- Infrared singularities



$$\rightarrow \infty$$

as $k^\mu \rightarrow 0$ (soft divergence)

or $k^\mu \parallel p^\mu$ (collinear divergence)

$$\frac{1}{(p-k)^2 - m^2} = \frac{1}{-2p \cdot k} \text{ (for } m = 0 \text{ or } m \neq 0)$$

$p \cdot k \rightarrow 0$ as

$k \rightarrow 0$ or $k^\mu \parallel p^\mu$ (for $m = 0$)

$k \rightarrow 0$ (for $m \neq 0$)

(Similar singularities also exist in virtual diagrams.)

- Solutions

Compute H_{ij} in pQCD in $n = 4 - 2\varepsilon$ dimensions

(dimensional regularization)

(1) $n \neq 4 \Rightarrow$ UV & IR divergences appear as $\frac{1}{\varepsilon}$ poles

in $\sigma_{ij}^{(1)}$ (Feynman diagram calculation)

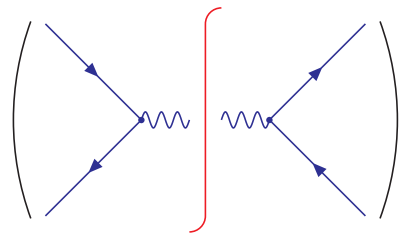
(2) H_{ij} is IR safe $\Rightarrow$ no $\frac{1}{\varepsilon}$ in H_{ij}

(H_{ij} is UV safe after "renormalization".)

Dimensional Regularization
(Revisit the Born Cross Section in n dimensions)

- $$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1}\, 2q_0} (2\pi)^n \cdot \delta^n (p_1 + p_2 - q) \cdot \overline{|m|^2}$$

- $$\overline{|m|^2} = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot$$



In n-dim, the polarization degree of freedom is (2) for a quark, and (n-2) for a gluon.

- Using the Naive-γ^5 prescription:

$$Tr\,[\not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L]\,(-1)$$
$$= Tr\,[\not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L]\,(-1) \qquad\qquad \gamma_\mu \not{p}_2 \gamma^\mu = -2\,(1-\varepsilon)\not{p}_2$$
$$= (-2)\,(1-\varepsilon)\,Tr\,[\not{p}_1 \not{p}_2 P_L]\,(-1)$$
$$= (-2)\,(1-\varepsilon) \cdot \frac{1}{2} \cdot 4\,(p_1 \cdot p_2)\,(-1)$$
$$= 2\,(1-\varepsilon)\,\hat{s}$$

- In n dimensions

$$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\varepsilon) \cdot \delta\,(1-\hat{\tau}) \equiv \sigma^{(0)} \cdot \delta(1-\hat{\tau})$$

Strong Coupling g in n dimensions

- In n dimensions

$$\int d^n x \mathcal{L}$$

$$\longrightarrow \int d^n x \left\{ \bar{\psi} i \not{\partial} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \cdots \right\}$$

The dimension in mass unit (μ)

$$[\psi] \sim \mu^{\frac{n-1}{2}}$$

$$[G] \sim \mu^{\frac{n-2}{2}}$$

$$\left[\bar{\psi} G \psi\right] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

Since $\left[g \bar{\psi} G \psi\right] \sim \mu^n$, so

$$[g] \sim \mu^{\frac{-n}{2} + 2} \qquad n = 4 - 2\varepsilon$$

$$= \mu^{\varepsilon}$$

$\Rightarrow$ g has a dimension in mass when $\varepsilon \neq 0$

$\Rightarrow$ Feynman rules should read $g \to g\mu^{\varepsilon}$

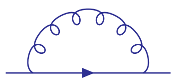
Calculations

- Tools needed for a NLO calculation are collected in Appendices A-D

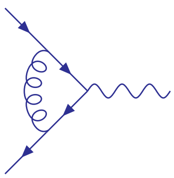
- The detailed calculation for each subprocess can be found in Appendices E

- In the following, I shall summarize the result for each subprocess

Virtual Corrections $(q\overline{q}')_{virt}$
(in Feynman Gauge)

-  $=$ 0

  $\frac{1}{\varepsilon_{IR}}$and $\frac{1}{\varepsilon_{UV}}$ poles cancel when $\varepsilon_{UV} = -\varepsilon_{IR} \equiv \varepsilon$

- 

  $\frac{1}{\varepsilon_{UV}}$ cancel $\Rightarrow$ Electroweak coupling is not renormalized by QCD interactions at one-loop order
  (Ward identity,
  a renormalizable theory)

  $\frac{1}{\varepsilon_{IR}}$ poles remain

- $\sigma^{(1)}_{virt}$ is free of ultraviolent singularity.

$$\sigma^{(1)}_{virt} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta\left(1 - \widehat{\tau}\right) \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon} \frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)} \cdot \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} \right\} \cdot (C_F)$$

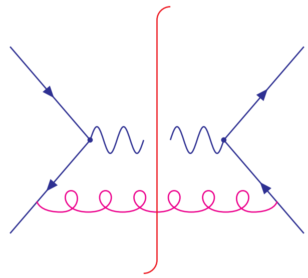
$-\frac{2}{\varepsilon^2}$: soft and collinear singularities

$-\frac{3}{\varepsilon}$: soft or collinear singularities

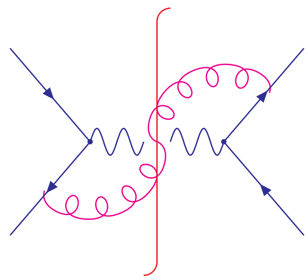
C_F: color factor

$\sigma^{(0)} \equiv \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\varepsilon)$

Real Emission Contribution $(q\overline{q}')_{real}$

- 

  $\sim \dfrac{1}{\varepsilon}$ Collinear

  $\sim \dfrac{1}{\varepsilon^2}$ Soft and Collinear

- $$\sigma^{(1)}_{\text{real}}\left(q\overline{q}'\right) = \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\varepsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau}\right\}$$

Note: $[\cdots]_+$ is a distribution,

$$\int_0^1 dz\, f(z)\left[\frac{1}{1-z}\right]_+ = \int_0^1 dz\frac{f(z)-f(1)}{1-z},$$ which is finite.

- In the soft limit, $\hat{\tau}\to 1$ $\left(\hat{\tau}=\frac{M^2}{\hat{s}}\right)$,

$$\sigma^{(1)}_{\text{real}}\left(q\overline{q}'\right) \longrightarrow \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta(1-\hat{\tau}) - \frac{4}{\varepsilon}\frac{1}{(1-\hat{\tau})_+} + 8\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right\}$$

$\left(q\bar{q}'\right)_{virt} + \left(q\bar{q}'\right)_{\text{real}}$ at NLO

-

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(1)}_{virt}\left(q\bar{q}'\right) + \sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right)$$

$$= \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$

$$\cdot\left\{\frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right.$$

$$\left. + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\hat{\tau})\right\}$$

Where we have used

$$\frac{-2}{\varepsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right] = \frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+$$

-

All the soft singularities $\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)$ cancel in $\sigma^{(1)}_{q\bar{q}'}$

$\Rightarrow KLN$ theorem

(Kinoshita-Lee-Navenberg)

-

$$\sigma^{(1)}_{q\bar{q}'} \sim \frac{1}{\varepsilon}(\text{term}) + \text{finite (terms)}$$

↑

Collinear Singularity

Factorization Theorem

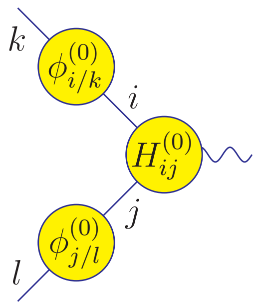
- Perturbative PDF

$\phi^{(0)}_{i/k} = \delta_{ik}\delta(1-x)$

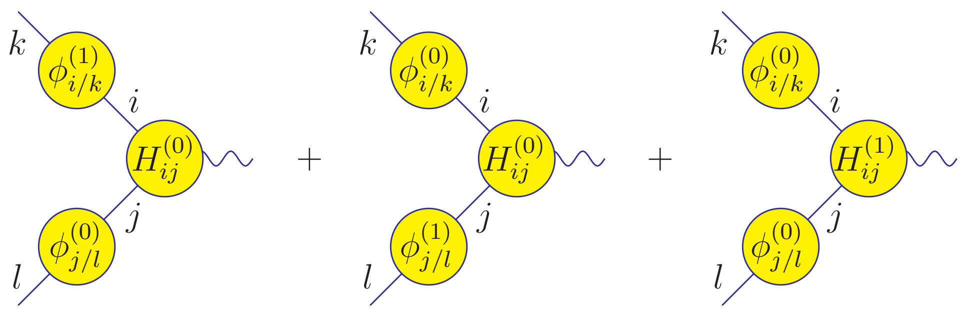
$\frac{\alpha_s}{\pi}\phi^{(1)}_{i/k}$ can be calculated from the definition of PDF.

(Process independent,but factorization scheme dependent)

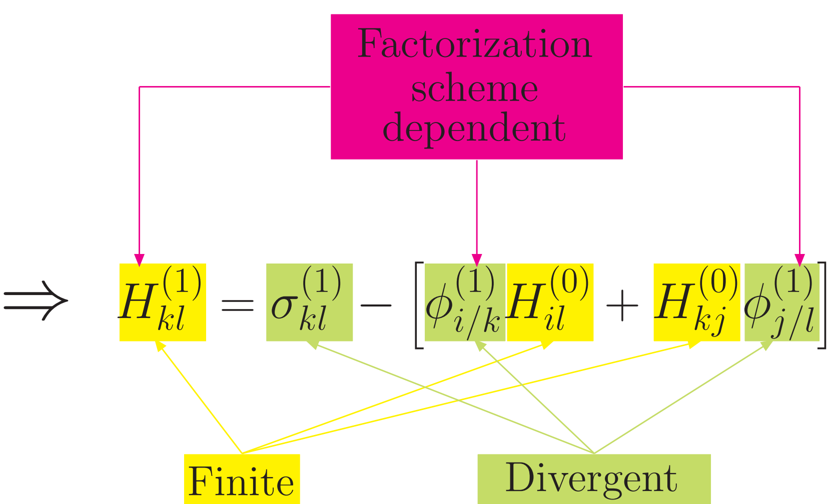
- (1)

$\sigma^{(0)}_{kl} =$  $\Rightarrow H^{(0)}_{kl} = \sigma^{(0)}_{kl}$

(2)

$\sigma^{(1)}_{kl} =$ 

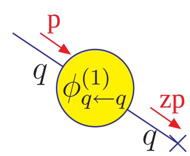
$$\Rightarrow H^{(1)}_{kl} = \sigma^{(1)}_{kl} - \left[\phi^{(1)}_{i/k} H^{(0)}_{il} + H^{(0)}_{kj}\phi^{(1)}_{j/l}\right]$$



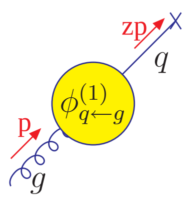
Perturbative PDF

- In $\overline{MS}$-scheme (modified minimal subtraction)

$$\phi^{(1)}_{q/q}(z) = \phi^{(1)}_{\bar{q}/\bar{q}}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow q}(z)$$
$$\phi^{(1)}_{q/g}(z) = \phi^{(1)}_{\bar{q}/g}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow g}(z)$$

where the splitting kernel for  is

$$P^{(1)}_{q\leftarrow q}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$
$$= C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right),$$

and for  is

$$P^{(1)}_{q\leftarrow g}(z) = T_R\left(z^2 + (1-z)^2\right),$$

where $C_F = \frac{4}{3}$ and $T_R = \frac{1}{2}$.

(Note: The Pole part in the $\overline{MS}$ scheme is

$$\frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon}(4\pi e^{-\gamma_E})^{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E$$

In the MS scheme, the pole part is just $\frac{1}{\varepsilon}$)

Find $H^{(1)}_{q\bar{q}'}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(0)} \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] \right.$$
$$\left. + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\}$$

-

$$H^{(1)}_{q\bar{q}'}(\hat{\tau}) = \sigma^{(1)}_{q\bar{q}'} - \left[2\phi^{(1)}_{q\leftarrow q}\sigma^{(0)}_{q\bar{q}'} \right]$$
$$= \hat{\sigma}^{(0)} \cdot \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \ln\left(\frac{M^2}{\mu^2}\right) \right.$$
$$\left. + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\}$$

where

$$\hat{\tau} = \frac{M^2}{\hat{s}} = \frac{M^2}{x_1 x_2 S}, \qquad \sigma^{(0)} = \hat{\sigma}^{(0)} \cdot (1-\varepsilon),$$
$$\hat{\sigma}^{(0)} = \frac{\pi}{12\hat{s}} g_w^2 = \frac{\pi g_w^2}{12S} \frac{1}{x_1 x_2}.$$

- pQCD prediction

$$\sigma_{hh'} = \left\{ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\sigma^{(0)}\delta(1-\hat{\tau})\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right) \right.$$
$$+ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{f\bar{f}}(\hat{\tau}) \right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right)$$
$$\left. + \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{fG}(\hat{\tau}) \right] \phi_{G/h'}\left(x_2,\mu^2\right) + (x_1 \leftrightarrow x_2) \right\}$$

Find $H_{qG}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{qG}^{(1)} = \sigma^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[\frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2 (1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] + \frac{1}{2} + 3\hat{\tau} - \frac{7}{2}\hat{\tau}^2 \right\}$$

-

$$H_{qG}^{(1)}(\hat{\tau}) = \sigma_{qG}^{(1)} - \left[\sigma_{q\bar{q}'}^{(0)} \phi_{\bar{q}'\leftarrow G}^{(1)}\right]$$

$$= \frac{\hat{\sigma}^{(0)}}{2} \cdot \left\{ P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[\ln\left(\frac{M^2}{\mu^2}\right) + \ln\left(\frac{(1-\hat{\tau})^2}{\hat{\tau}}\right) \right] + \frac{1}{4} + \frac{3}{2}\hat{\tau} - \frac{7}{4}\hat{\tau}^2 \right\}$$

- Similarly,

$$H_{G\bar{q}'}^{(1)} = \sigma_{G\bar{q}'}^{(1)} - \left[\phi_{q\leftarrow G}^{(1)} \sigma_{q\bar{q}'}^{(0)}\right]$$

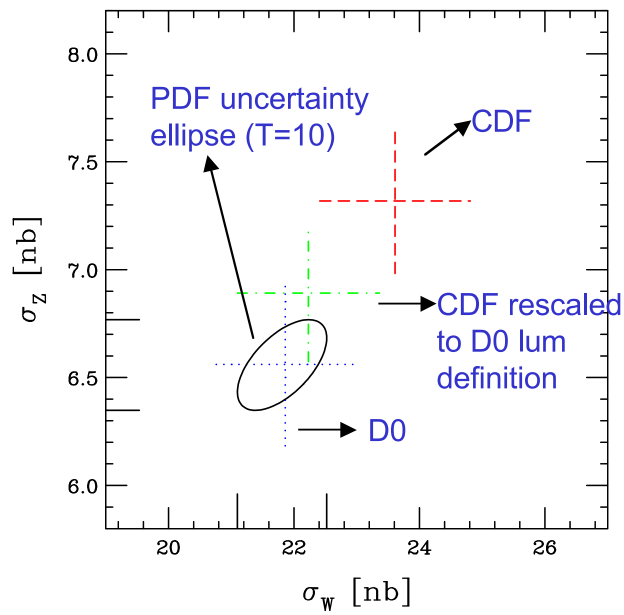
$$= H_{qG}^{(1)}$$

(Note: If we choose the renormalization scale $\mu^2 = M^2$, then $\ln\left(\frac{M^2}{\mu^2}\right) = 0$)

W and Z production

* CDF and D0 would like to use their W and Z cross sections for luminosity determination
* D0 cross sections close to center of PDF prediction ellipse; not the case with CDF

MSU study



J. Pumplin, D. Stump, R. Brock, D. Casey, J. Huston, J. Kalk, H.L. Lai, W.K. Tung: hep-ph/0101051

Summary

- $\phi_{f/h}(x,\mu^2)$ depends on scheme ($\overline{MS}$,DIS,...)
  $\Rightarrow H_{ij}$ **scheme dependent**

- Evolution equations allow us to perdict
  q^2-**dependent of** $\phi(x,q^2)$

- Essentially identical procedure for
  $hh' \rightarrow jets$, inclusive $Q\overline{Q}$,...
  But, when the Born level process involves
  strong interaction (eg. $q\overline{q} \rightarrow t\bar{t}$),
  it is also necessary to renormalize the
  strong coupling α_s, etc, to eliminate
  ultraviolate singularities

Appendix A
γ-matrices in n dimensions

- Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\}_+ \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$
$$\mu, \nu = 1, 2, \cdots, n \qquad g^{\mu\nu} = diag\,(1, -1, \cdots, -1)$$
$$g^{\mu\nu} g_{\mu\nu} = n$$
$$\left\{\gamma^\mu, \gamma^5\right\}_+ = 0 \qquad \left(\text{Naive-}\gamma^5\text{prescription}\right)$$

This works in calculating the inclusive rate of W-boson , but fails in the differential distributions of the leptons from the W-boson decay.

- Matrix identities

$n = 4 - 2\varepsilon$

$$\gamma_\mu \not{a}\gamma^\mu = -2\,(1-\varepsilon)\not{a}$$
$$\gamma_\mu \not{a}\not{b}\gamma^\mu = 4a\cdot b - 2\varepsilon\not{a}\not{b}$$
$$\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} + 2\varepsilon\not{a}\not{b}\not{c}$$

- Traces

$$Tr\,[\not{a}\not{b}] = 4\,(a\cdot b)$$
$$Tr\,[\not{a}\not{b}\not{c}\not{d}] = 4\,\{(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)\}$$
$$Tr\,[\gamma_5\not{a}\not{b}] = 0$$

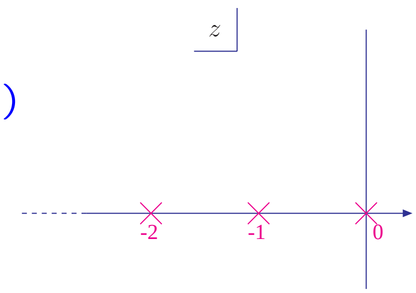
Appendix B
Some integrals and "special functions"

- The "Gamma function"

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \qquad (\mathrm{Re}(z) > 0)$$

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} \quad \text{(for all z)}$$

$\Rightarrow$ Poles in $\Gamma(z)$



$$\Gamma(n) = (n-1)! \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \cdots$$

$(\gamma_E = 0.5772\cdots,$ Euler constant$)$

$$\Gamma(1-\varepsilon) = -\varepsilon\Gamma(\varepsilon) = 1 + \varepsilon\gamma_E + \frac{1}{2}\varepsilon^2\left(\frac{\pi^2}{6} + \gamma_E^2\right) + O\left(\varepsilon^3\right)$$

$$\Gamma(1-\varepsilon)\Gamma(1+\varepsilon) = 1 + \varepsilon^2\frac{\pi^2}{6} + O\left(\varepsilon^4\right)$$

$$z^\varepsilon = e^{\ln z^\varepsilon} = e^{\varepsilon \ln z} = 1 + \varepsilon \ln z + \cdots$$

- The "Beta function"

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$B(\alpha,\beta) = \int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1} = \int_0^\infty dy\, y^{\alpha-1}(1+y)^{-\alpha-\beta}$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta\, (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}$$

- Feynman trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax + b(1-x)]^2}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}}$$

- n-dimension integrals

$$\int d^n l \frac{l_\mu}{(l^2 - M^2)^\alpha} = 0$$

$$\int d^n l \frac{l_\mu l_\nu}{(l^2 - M^2)^\alpha} = \int d^n l \frac{\left(\frac{l^2 g_{\mu\nu}}{n}\right)}{(l^2 - M^2)^\alpha}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 - M^2)^\alpha} = i \frac{(-1)^\alpha}{(4\pi)^{n/2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \left(\frac{1}{M^2}\right)^{\alpha - \frac{n}{2}}$$

$$\int d^n l \frac{l^2}{(l^2 - M^2)^\alpha} = \int d^n l \frac{\left(l^2 - M^2\right) + M^2}{(l^2 - M^2)^\alpha}$$

- $$\mathrm{Re}\left[(-1)^\varepsilon\right] = 1 - \varepsilon^2 \frac{\pi^2}{2} + O\left(\varepsilon^4\right)$$

- "plus distribution" — to isolate $\frac{1}{\varepsilon}$ poles

Consider $\dfrac{1}{(1-z)^{1+2\varepsilon}}$

$$= \frac{1}{(1-z)^{1+2\varepsilon}} - \left[\delta(1-z)\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} + \frac{1}{2\varepsilon}\delta(1-z)\right]$$

cancel

because $\displaystyle\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$ for $\varepsilon \to 0^-$

$$\equiv \left[\frac{1}{(1-z)^{1+2\varepsilon}}\right]_+ - \frac{1}{2\varepsilon}\delta(1-z)$$

$$= \frac{1}{(1-z)_+} - 2\varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + O\left(\varepsilon^2\right) - \frac{1}{2\varepsilon}\delta(1-z)$$

because $\displaystyle\frac{1}{(1-z)^{2\varepsilon}} = (1-z)^{-2\varepsilon}$

$$= 1 - 2\varepsilon\ln(1-z) + \cdots$$

- $[\cdots]_+$ is a distribution

$$\int_0^1 dz\, f(z)\left[\frac{1}{1-z}\right]_+$$

$$\equiv \int_0^1 dz\,\frac{f(z)}{1-z} - \int_0^1 dz\, f(z)\,\delta(1-z)\int_0^1 \frac{dz'}{(1-z')}$$

$$= \int_0^1 dz\frac{f(z)-f(1)}{1-z},$$ which is finite.

Appendix C
Angular integrals in *n* dimensions

- In n dimensions

$$\int d^n x = \int r^{n-1} d\Omega_{n-1}$$

-

$$\int d\Omega_n = \int_0^{\pi} d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int_0^{\pi} d\theta_{n-2} \sin^{n-2}\theta_{n-2} \cdots \int_0^{\pi} d\theta_1 \sin\theta_1 \int_0^{2\pi} d\phi$$

$$\Rightarrow \int d\Omega_1 = \int_0^{2\pi} d\phi \qquad \longrightarrow \Omega_1 = 2\pi$$

$$\int d\Omega_2 = \int_0^{\pi} d\theta_1 \sin\theta_1 \int d\Omega_1 \qquad \longrightarrow \Omega_2 = 4\pi$$

$$\vdots$$

$$\int d\Omega_n = \int_0^{\pi} d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int d\Omega_{n-1}$$

$$\Rightarrow \quad \Omega_n = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}$$ from repeated use of $B(\alpha, \beta)$

$$= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$ because $\Gamma(n) = \frac{2^{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

Appendix D
Two-particle phase space in n dimensions

- $$\int_{PS_2(p)} dk\,dq = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}\,2k_0}\frac{d^{n-1}\vec{q}}{(2\pi)^{n-1}\,2q_0}\cdot(2\pi)^n\,\delta^n\,(p-q-k)$$

  with $k^\mu = \left(k_0, \vec{k}\right)$, etc.

  Use $\frac{d^{n-1}\vec{q}}{2q_0} = \int d^n q \delta^+\left(q^2 - Q^2\right)$, we get

- $$\int_{PS_2(p)} dk\,dq = \frac{1}{(2\pi)^{n-2}}\int \frac{d^{n-1}\vec{k}}{2k_0}\delta^+\left((p-k)^2 - Q^2\right)$$
  $$= \frac{1}{(2\pi)^{n-2}}\int \frac{dk\,k^{n-3}}{2}\int d\Omega_{n-2}\delta\left(\hat{s} - 2k\sqrt{\hat{s}} - Q^2\right)$$
  $$\left(p^2 \equiv \hat{s}, k^2 = 0, k = \left|\vec{k}\right|\right)$$

  Use $n = 4 - 2\varepsilon$, then in the c.m. frame $\left(p^\mu = \left(\sqrt{\hat{s}}, \vec{0}\right)\right)$,

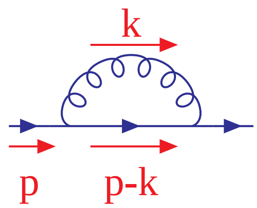
- $$\int_{PS_2(p)} dk\,dq = \frac{\Omega_{n-3}}{(2\pi)^{2(1-\varepsilon)}}\int \frac{dk\,k^{1-2\varepsilon}}{4\sqrt{\hat{s}}}\int_0^\pi d\theta\,(\sin\theta)^{1-2\varepsilon}\cdot\delta\left(k - \frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}\right)$$

  Use new variables:

  $$z = \frac{Q^2}{\hat{s}}, y = \frac{1}{2}(1+\cos\theta) \Rightarrow k = \frac{\sqrt{\hat{s}}}{2}(1-z),$$

- $$\int_{PS_2(p)} dk\,dq = \frac{1}{8\pi}\left(\frac{4\pi}{Q^2}\right)^\varepsilon \frac{z^\varepsilon(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^1 dy\,[y(1-y)]^{-\varepsilon}$$

Appendix E
Explicit Calculations

- Consider 

$$\int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu (\not p - \not k) \gamma^\mu}{(k^2 + i\epsilon)\left((p-k)^2 + i\epsilon\right)}$$

$$\to \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{(2-n)(\not p - \not k)}{\left[k^2 - 2k \cdot xp\right]^2} \qquad (l \equiv k - xp)$$

$$= \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left[(1-x)\not p - \not l\right]}{\left[l^2 + i\epsilon\right]^2}$$

$$= \left[\left(1 - \frac{n}{2}\right)\not p\right] \cdot \underbrace{\int \frac{d^n l}{(2\pi)^n} \frac{1}{\left[l^2 + i\epsilon\right]^2}}$$

$$\downarrow$$

$$= 0 \quad \left(\begin{array}{c}\text{Because there is} \\ \text{no mass scale}\end{array}\right)$$

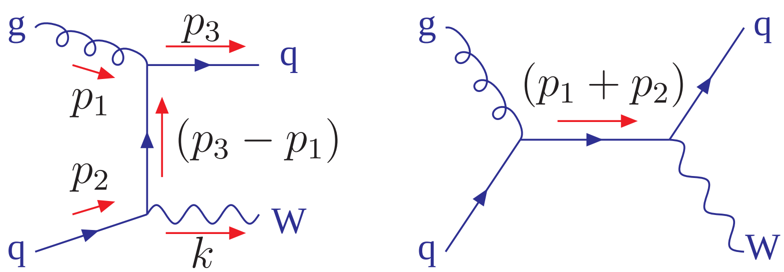
$$\uparrow$$

$$\left(\begin{array}{c}\text{Due to cancellation} \\ \text{of } \frac{1}{\varepsilon_{UV}} \text{ and } \frac{1}{\varepsilon_{IR}} \\ \text{Trick:} A = A - B + B\end{array}\right)$$

$$= \int \frac{d^n l}{(2\pi)^n} \left\{ \underbrace{\left[\frac{1}{(l^2)^2} - \frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\text{IR div.}} + \underbrace{\left[\frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\text{UV div.}} \right\}$$

$$= \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{IR}}\right) + \frac{i}{16\pi}\left(\frac{1}{\varepsilon_{UV}}\right), \qquad \left(\begin{array}{c} n - 4 = 2\varepsilon_{IR} \\ 4 - n = 2\varepsilon_{UV}\end{array}\right)$$

- consider the real emission process



Define the Mandelstam variables

$$\hat{s} = (p_1 + p_2)^2 = \; 2p_1 \cdot p_2$$
$$\hat{t} = (p_1 - p_3)^2 = -2p_1 \cdot p_3$$
$$\hat{u} = (p_2 - p_3)^2 = -2p_2 \cdot p_3$$

After averaging over colors and spins

$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{2(1-\varepsilon)}\frac{1}{2}\right)}_{\text{Spin}} \cdot \underbrace{\left(\frac{1}{3}\cdot\frac{1}{8}\right)\cdot Tr\left(t^a t^a\right)}_{\text{Color}} \cdot (g\mu^{\varepsilon})^2$$

$$\cdot\, g_w^2 \cdot 2(1-\varepsilon)$$

$$\cdot \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}}\right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon\right]$$

Note: The d.o.f. of gluon polarization is $2(1-\varepsilon)$, and that of quark polarization is 2.

- In the parton c.m. frame, the constituent cross section

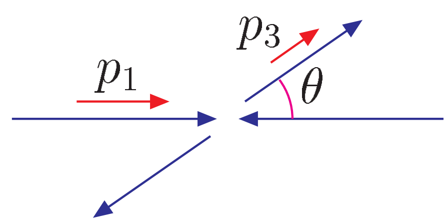
$$\hat{\sigma} = \frac{1}{2\hat{s}}\overline{|\mathcal{M}|^2}\cdot(PS_2)$$

$$= \frac{1}{2\hat{s}}\cdot\left\{\frac{1}{4}\cdot\frac{1}{6}\cdot 2g_s^2\mu^{2\varepsilon}g_w^2(1-\varepsilon)\cdot\right.$$

$$\left.\left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}}-\frac{\hat{t}}{\hat{s}}\right)-\frac{2\hat{u}M^2}{\hat{t}\hat{s}}+2\varepsilon\right]\right\}$$

$$\cdot\left\{\frac{1}{8\pi}\left(\frac{4\pi}{M^2}\right)^{\varepsilon}\frac{1}{\Gamma(1-\varepsilon)}\hat{\tau}^{\varepsilon}(1-\hat{\tau})^{1-2\varepsilon}\int_0^1 dy\,[y(1-y)]^{-\varepsilon}\right\}$$

where $\quad y \equiv \frac{1}{2}(1+\cos\theta)$



Using $\quad \hat{t} = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)(1-y)$

$$\hat{u} = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)y$$

and

$$\int_0^1 dy\,y^{\alpha}(1-y)^{\beta} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)},$$

we get

$$\hat{\sigma}_{qG} = \hat{\sigma}^{(0)}\frac{\alpha_s}{4\pi}\cdot\left\{2P^{(1)}_{q\leftarrow g}(\hat{\tau})\left[\frac{-1}{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}+\ln\frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}}\right]\right.$$

$$\left.+\frac{3}{2}+\hat{\tau}-\frac{3}{2}\hat{\tau}^2\right\},$$

with

$$P^{(1)}_{q\leftarrow g}(\hat{\tau}) = \frac{1}{2}\left[\hat{\tau}^2+(1-\hat{\tau})^2\right]$$

$$\hat{\sigma}^{(0)} \equiv \frac{\pi}{12}g_w^2\frac{1}{\hat{s}}$$